Math 504: Modern Algebra, Fall Quarter 2017
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Homework 7
Due: Monday, November 20

Problem 7.1. Let k be a field and $T\colon V \to V$ a linear transformation of a finite dimensional vector space V over k. Recall that the *minimal polynomial* of T is the monic polynomial $m_T(x) \in k[x]$ of smallest degree satisfying $m_T(T) = 0$.

(1) For $g \in k[x]$, show that m_T divides g if and only if $g(T) = 0$.
(2) Viewing V as a $k[x]$-module where x acts via T, show that the ideal $\operatorname{Ann}_{k[x]}(V)$ is generated by m_T.

Problem 7.2. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

(1) Find a matrix P such that PAP^{-1} is in rational canonical form.
(2) Find a matrix P such that PAP^{-1} is in Jordan canonical form.

Problem 7.3.

(1) Determine all possible rational canonical forms for a linear transformation with characteristic polynomial $x^2(x^2+1)^2$.
(2) Determine up to similarity all matrices $A \in M_2(\mathbb{Q})$ of order 4.
(3) Determine up to similarity all matrices $A \in M_2(\mathbb{C})$ of order 4.

Problem 7.4. Recall that a ring R is called *left-Noetherian* (resp. *right-Noetherian*) if every ascending chain of left ideals (resp. right ideals) terminates.

Define the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in \mathbb{Z}; b, c \in \mathbb{Q} \right\} \subset M_2(\mathbb{Q})$$

Show that R is right Noetherian but not left Noetherian.

Problem 7.5. Recall that a ring R is called *Noetherian* if it is both left- and right-Noetherian, and that R is called *Artinian* if is both left and right-Artinian, where *left-Artinian* (resp. *right-Artinian*) means that every descending chain of left ideals (resp. right ideals) terminates.

(1) Let k be a field and R a k-algebra (this means that R is ring together with a ring homomorphism $k \to R$). If R is finite dimensional as a k-vector space, show that R is both Noetherian and Artinian.
(2) If R is a PID and $I \subset R$ is a non-zero ideal, show that R/I is both Noetherian and Artinian.

Problem 7.6. Let D is a division ring and define $R = M_n(D)$.

(1) Show that R is a simple ring (i.e. R has no two-sided ideals other than 0 and R).
(2) For any $k = 1, \ldots, n$, show that the left ideal $I_k \subset R$ of matrices where all entries are zero except possibly entries in the kth column is a simple left R-module (i.e. I_k contains no left R-submodules other than 0 and I_k).