Math 504: Modern Algebra, Fall Quarter 2017
Jarod Alper
Homework 6
Due: Monday, November 13

Problem 6.1.

(1) Prove that $x^3 + nx + 2 \in \mathbb{Z}[x]$ is irreducible if and only if $n \neq 1, -3, -5$.
(2) Factor $x^8 - 1$ over each of the following rings: (a) $\mathbb{Z}$, (b) $\mathbb{Z}/2$ and (c) $\mathbb{Z}/3$.

Problem 6.2.

(1) Let R be a commutative ring. Show that $R^n \cong R^m$ are isomorphic as R-modules if and only if $n = m$.
(2) Let k be a field and V be an infinite dimensional vector space with a countable basis. Let R be the ring $\mathrm{End}_k(V)$ of k-linear endomorphisms. Show that $R^2 \cong R$ as (left) R-modules.

Problem 6.3. Let R be a ring and M', M and M'' be R-modules. Let $f\colon M' \to M$ and $g\colon M \to M''$ be R-module homomorphisms. We say that

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a *short exact sequence* if f is injective, g is surjective and $\mathrm{im}(f) = \ker(g)$. (In other words, this condition is requiring that f is injective and that $M'' \cong M/M'$ where we have identified M' with the image submodule $\mathrm{im}(f) \subset M$, or equivalently that g is surjective and $M' \cong \ker(g)$.)

(1) Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of R-modules. Show that M is Noetherian if and only if M' and M'' are Noetherian.
(2) Let R be a Noetherian ring. Show that any finitely generated R-module is a Noetherian R-module.

Problem 6.4. Recall that if M is an R-module, then (1) $\mathrm{Tor}_R(M) = \{m \mid$ there exists $r \in R$ such that $rx = 0\}$, (2) M is *torsion* if $M = \mathrm{Tor}_R(M)$ and (3) the *rank* $\mathrm{rk}(m)$ *of* M is the maximum number of R-linearly independent elements of M.

Let R be an integral domain.

(1) Show that $\mathrm{rk}(M) = n$ if and only if there exists a submodule $R^n \subset M$ such that M/R^n is torsion.
(2) For R-modules M and M', show that $\mathrm{rk}(M \oplus M') = \mathrm{rk}(M) + \mathrm{rk}(M')$.
(3) For R-modules M and M', show that $\mathrm{Tor}_R(M \oplus M') \cong \mathrm{Tor}_R(M) \oplus \mathrm{Tor}_R(M')$.

Problem 6.5. Let R be an integral domain. Recall that if M is an R-module, then $\mathrm{Ann}(M) = \{r \in R \,|\, rm = 0 \text{ for all } m \in M\}$.

(1) If M is a finitely generated R-module, show that if M is torsion, then $\mathrm{Ann}(M) \neq 0$.
(2) Give a counterexample to the above statement if M is not finitely generated.

Problem 6.6. Suppose that R is a commutative ring such that every submodule of a free module is also free. Show that R is a PID.