Math 504: Modern Algebra, Fall Quarter 2017
Jarod Alper
Homework 5
Due: Monday, October 30

**Problem 5.1.**

(1) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group of order 8 and $\mathbb{R}[Q_8]$ be the group algebra of $Q_8$ over $\mathbb{R}$. Let $a = e_1 + e_{-1} \in \mathbb{R}[Q_8]$. Show that $a\mathbb{R}[Q_8] = \mathbb{R}[Q_8]a$. Conclude that $a\mathbb{R}[Q_8]$ is a two-sided ideal and that there is an isomorphism of $\mathbb{R}[Q_8]/a\mathbb{R}[Q_8] \cong \mathbb{H}$, where $\mathbb{H}$ denotes the Quarternion algebra.

(2) Find an element $a \in \mathbb{C}[S_3]$ such that the left ideal $a\mathbb{C}[S_3]$ is not equal to the right ideal $\mathbb{C}[S_3]a$.

**Problem 5.2.** Classify all ring homomorphisms $M_2(\mathbb{Z}) \to \mathbb{Z}$.

**Problem 5.3.** Let $R$ be a commutative ring.

(1) Show that an ideal $I \subset R$ is prime if and only if $R/I$ is an integral domain.
(2) Show that an ideal $I \subset R$ is maximal if and only if $R/I$ is a field.
(3) Let $\mathfrak{p} \subset R$ be a prime ideal. Show that the prime ideals in the localization $R_{\mathfrak{p}}$ are in bijective correspondence to the prime ideals in $R$ contained in $\mathfrak{p}$.

**Problem 5.4.** Let $\mathbb{Z}[i]$ be the subring of $\mathbb{C}$ consisting of complex numbers $a + bi$ where $a, b \in \mathbb{Z}$. (The ring $\mathbb{Z}[i]$ is called the ring of *Gaussian integers*.).

(1) Show that $\mathbb{Z}[i]$ is a Euclidean domain. This implies that $\mathbb{Z}[i]$ is a PID and a UFD.
(2) Find a factorization of 6 as a finite product of irreducible elements of $\mathbb{Z}[i]$.

**Problem 5.5.** Let $\mathbb{Z}[\sqrt{-5}]$ be the subring of $\mathbb{C}$ consisting of complex numbers $a + b\sqrt{-5}$ where $a, b \in \mathbb{Z}$.

(1) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.
    *Hint:* Find two factorizations of 9 in $\mathbb{Z}[\sqrt{-5}]$.
(2) Find an ideal in $\mathbb{Z}[\sqrt{-5}]$ that is not principal.

**Problem 5.6.** Find all of the ideals of the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.