Math 504: Modern Algebra, Fall Quarter 2017
Jarod Alper
Homework 2
Due: Monday, October 9

Problem 2.1. Prove the uniqueness part of the Jordan–Hölder theorem. That is, if G is a finite group and

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = G \quad \text{and} \quad 1 = N'_0 \trianglelefteq N'_1 \trianglelefteq \cdots \trianglelefteq N'_s = G$$

are composition series of G, then $r = s$, and there is permutation $\sigma \in S_r$ and isomorphisms $N_i/N_{i+1} \cong N'_{\sigma(i)}/N'_{\sigma(i)+1}$.

Hint: Handle first the case when $s = 2$ and then use induction on $\min\{r, s\}$ by considering $N_{r-1} \cap N'_{s-1}$.

Problem 2.2. A group G is called *solvable* if there is chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G \tag{2.1}$$

such that for $i = 1, \ldots, r-1$, $G_i \trianglelefteq G_{i+1}$ is a normal subgroup and G_{i+1}/G_i is abelian. Prove that the following are equivalent for a finite group G:

(a) G is solvable;
(b) There exists a chain of subgroups as in (2.1) such that for $i = 1, \ldots, r-1$, $G_i \trianglelefteq G_{i+1}$ is a normal subgroup and G_{i+1}/G_i is cyclic; and
(c) There exists a chain of subgroups as in (2.1) such that for $i = 1, \ldots, r-1$, $G_i \trianglelefteq G_{i+1}$ is a normal subgroup and $G_{i+1}/G_i \cong \mathbb{Z}/p_i$ for a prime p_i.

Problem 2.3.

(1) Show that $\mathrm{Aut}(\mathbb{Q}) \cong \mathbb{Q}^\times$.
(2) Show that the subgroup

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{Q}^\times, b \in \mathbb{Q} \right\} \subset \mathrm{GL}_2(\mathbb{Q})$$

is isomorphic to a semi-direct product $\mathbb{Q} \rtimes_\phi \mathbb{Q}^\times$ for a suitable choice of ϕ.
(3) Show that $\mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes_\phi \mathbb{Z}/2$ for a suitable choice of ϕ.

Problem 2.4.

(1) Classify finite abelian groups of order 72.
(2) Compute the center of S_n.
(3) Show that any finite group is isomorphic to a subgroup of the alternating group A_n for some n.

Problem 2.5.

(1) Find a Jordan–Hölder composition series for S_4.
(2) Find a Jordan–Hölder composition series for $\mathrm{GL}_2(\mathbb{F}_3)$.
(3) Are $\mathrm{SL}_2(\mathbb{F}_3)$ and S_4 isomorphic?

Bonus Problem 2.6. Let $n \geq 5$. Prove that A_n is a simple group as follows:

(1) Show that A_n is generated by 3-cycles (i.e., permutations of the form (ijk) with i, j, k distinct).
(2) Show that if $N \trianglelefteq A_n$ is a normal subgroup containing a 3-cycle, then $N = A_n$.

(3) Prove that A_n is a simple group by considering the following cases for a normal subgroup $N \trianglelefteq A_n$:
- N contains a 3-cycle;
- N contains an element that is the product of disjoint cycles, at least one of which has length ≥ 4;
- N contains an element that is the product of disjoint cycles, at least two of which have length 3;
- N contains an element that is the product of a 3-cycle and some 2-cycles; and
- Every element of N is the product of an even number of 2-cycles.