Math 504: Modern Algebra, Fall Quarter 2017
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Homework 1
Due: Monday, October 2

Problem 1.1. Summarize your exposure to algebra. What courses did you take as an undergrad? What have you studied on your own? Looking at the syllabus, what topics have you not been exposed to at all? Were there any topics that you found particularly challenging in algebra the first time you learned them? What are your expectations from this course?

Problem 1.2. Show that the group of orientation-preserving symmetries of the cube[1] is isomorphic to S_4.

Hint: Consider the four diagonals of the cube.

Problem 1.3. Recall that the *alternating group* A_n is defined as the subgroup of the symmetric group S_n consisting of even permutations.

(1) Show that A_4 is not isomorphic to the dihedral group D_{12}.
(2) Write down the lattice of subgroups for A_4; that is, determine each subgroup of A_4 and then draw a graph where the vertices correspond to the subgroups and there is a line between two vertices if there is a containment.

Problem 1.4.

(1) Find inclusions of groups $F \subset G \subset H$ such that $F \subset G$ and $G \subset H$ are normal subgroups, but $F \subset H$ is not.
(2) Find an example of a group G such that every element of G has finite order but such that G is not finite.

Problem 1.5. Let p be a prime and recall that $\mathbb{F}_p := \mathbb{Z}/p$. Consider the set $G \subset \mathrm{GL}_3(\mathbb{F}_p)$ consisting of upper triangular 3×3 matrices with 1's along the diagonal.

(1) Show that G is a subgroup.
(2) Determine the center $Z(G)$ of G.
(3) Determine $G/Z(G)$.

Problem 1.6. Determine all groups of order 6. (You are not allowed to use Sylow's theorems but you may use Lagrange's theorem.)

Problem 1.7. Let $n > 1$ be an integer.

(1) Find an injective homomorphism $S_n \to \mathrm{GL}_n(\mathbb{R})$.
(2) Determine whether the image subgroup is normal.
(3) For $n = 2$, determine its normalizer.

[1] To be more precise, by an orientation-preserving symmetry, we mean a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ which (1) preserves distance, (2) has determinant 1 and (3) sends the cube with vertices $(\pm 1, \pm 1, \pm 1)$ to itself. Sometimes these are called rigid motions or orientation-preserving rigid motions.