# POSSIBLE GROUP PROJECTS

Math 480A: Algebraic Complexity Theory
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*There can be more than one group addressing the same project. In most of the projects below, there are many possible directions and subtopics you can consider.*

**Project 1.** In lecture, we sketched using Gaussian elimination that $\det_n \in VP$. Fill in the details by showing that this algorithm can be realized on an arithmetic circuit. See §6 of Landsberg's "Geometry and Complexity Theory."

**Project 2: Relationship between the questions $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ and $\mathbf{VP} \stackrel{?}{=} \mathbf{VNP}$.** Suppose you could show that P=NP. Does this imply that VP=VNP? What about the converse? Possible references:

- Chapter 4 of Burgisser, "Completeness and Reduction in Algebraic Complexity Theory."
- Peter Burgisser, "Cook's versus Valiant's hypothesis", Theor. Comp. Sci., 235:71?88, 2000.

**Project 3: Immanants.** Immanants are polynomials that are generalizations of the permanent and determinant. Immanants provide many more examples of sequences of polynomials that allow us to try to interpolate between the determinant and permanent. If one can understand the computational complexity of immanants, this could shed some light on what makes the permanent hard and the determinant easy. In this project, you should explore the computation complexity of different immanants. Possible reference: Chapter 7 of Burgisser, "Completeness and Reduction in Algebraic Complexity Theory."

**Project 4: Arithmetic/determinant complexity of low degree polynomials.** Study the arithmetic and/or determinant complexity of homogeneous polynomial $f(x_1, \ldots, x_n)$ where $d$ and/or $n$ are fixed small numbers. Possible subtopics:

- Consider binary forms of degree $d$ (i.e. homogeneous polynomials of degree $d$ in $x$ and $y$). What is the arithmetic complexity and/or determinant complexity of a binary form? You are free to focus on either the arithmetic complexity or the determinant complexity.
- Same question for homogenous polynomials of degree 3 in three variables or degree 3 in 4 variables.
- In your situation, try to show that the locus of all polynomials with fixed arithmetic (or determinantal) complexity is not closed; that is, there is a family of polynomials $f_t(x_1, \ldots, x_n)$ varying continuously on a number $t$ such that $\mathrm{rk}(f_0) \neq \mathrm{rk}\, f_t$.

**Project 5: Waring rank.** A central question in algebraic complexity theory concerns the Waring rank. The *Waring* rank of a homogenous polynomial $f(x_1, \ldots, x_n)$ of degree $d$ is the smallest integer $r = \mathrm{rk}(f)$ so that $f$ can be written as $L_1^d + \cdots + L_k^d$ where each $L_i(x_1, \ldots, x_n)$ is linear. Here are some possible subtopics you could investigate:

- What is the Waring rank of a general (or random) homogenous polynomial $f(x_1, \ldots, x_n)$ of degree $d$?
- Consider the locus $W_r = \{f \in k[x_1, \ldots, x_n]_d \mid \mathrm{rk}(f) = r\}$. Show that this locus can be defined using the Segre embedding of the Veronese.
- Compute the locus $W_r$ for small $d$ and $n$. For example, you could fix $n = 2$ and consider arbitrary $d$. Or you could take $n = d = 3$.
- Explore the relationship with apolarity (requires more algebra background).
- What is the Waring rank of the determinant?
- Explore the Strassen's conjecture or Comon's conjecture.

**Project 6: Symmetries of $\det_n$ and $\mathrm{perm}_n$.** The *symmetric group* of a polynomial $f(x_1, \ldots, x_n)$ is defined as the set of invertible $n \times n$ matrices $A$ such that $f(Ax) = f(x)$ for all $x$ (where $Ax$ denotes the multiplication of the matrix $A$ by the vector $x$). Explore the symmetry groups of $\det_n$ and $\mathrm{perm}_n$. These were first determined by Frobenius (1892) and Marcus–May (1962). Can you use this to show $\mathrm{dc}(\mathrm{perm}_n) > n$?

**Project 7: Matrix multiplication.** How fast can a computer multiply two $n \times n$ matrices. The naive algorithm takes $O(n^3)$ steps. Is this the best possible way? Some possible subtopics:

- Show Strassen's formula for showing that two $2 \times 2$ matrices can be multiplied using only 7 multiplications. Use this fact to develop an algorithm for matrix multiplication with running time $O(n^{\log_2 7})$.
- Explore the conjecture that the running time of matrix multiplication is $O(n^{2+\epsilon})$ for any $\epsilon > 0$. What is the state of the art?
- What is the fewest number of multiplications needed to multiply two $3 \times 3$ matrices? $4 \times 4$?

**Project 8: Oracles in algebraic complexity theory.** Investigate the use of oracles in Turing machines in complexity theory. Show that there are some oracles where P=NP and others were P $\neq$ NP. See Sipser §6.3. Is there an analogous notion for arithmetic complexity of polynomials?

For that matter, you could consider any other advanced topic in classical complexity theory and try to see for yourself if there are analogous results in algebraic complexity theory.

**Project 9: Geometric complexity theory.** Show that Valiant's conjecture can be formulated using group actions and orbits as follows. Let $\mathrm{GL}_n$ be the group of invertible $n \times n$ matrices. This group acts on the vector space $k[x_1, \ldots, x_n]_d$ of degree $d$ homogenous polynomials $f(x_1, \ldots, x_n)$. Valiant's conjecture is equivalent to the statement: For every integer $k \geq 1$, the orbit closure

$$\overline{\mathrm{GL}_n \cdot (x_{m,m}^{m-n} \,\mathrm{perm}_n)}$$

where $m = n^k$ does not contain $\det_m$ for sufficiently large $n$. You can try possibly to further translate this question into representation theory.

**Project 10: Undecidability of polynomials.** Consider the language HILB consisting of all polynomials $f(x_1, \ldots, x_n)$ with integral coefficients such that $f$ has an integral root (i.e., there exists integers $a_1, \ldots, a_n$ with $f(a_1, \ldots, a_n) = 0$). Hilbert's 10th problem at the turn of the 20th century asks whether HILB is deciable, and it was shown 70 years later by Matiyasevich that HILB is not decidable. In this project, you could explore this problem and try to under Matiyasevich's argument.