# VNP
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Sequence $\{f_n\}$ of polynomials $f_n(x_1, \dots, x_{m_n})$ where
$m_n = \#$ variable
$\deg(f_n)$ = degree of $f_n$

Def We say $\{f_n\}$ is p-computable if
(1) $m_n$
(2) $\deg(f_n)$
(3) $C(f_n)$
are poly-banded.

$VP = \{ \{f_n\} \text{ p-computable} \}$

Prop: $\{DET_n\} \in VP$
We dont know if $\{PERM_n\} \in VP$.

VARIANT { Use expression size instead of arith. complexity }

$VP_e = \{ \{f_n\} \mid$ (1). # variables of $f_n$ is poly banded (2). $\deg(f_n)$ (3). $C_e(f_n)$ is poly banded $\}$

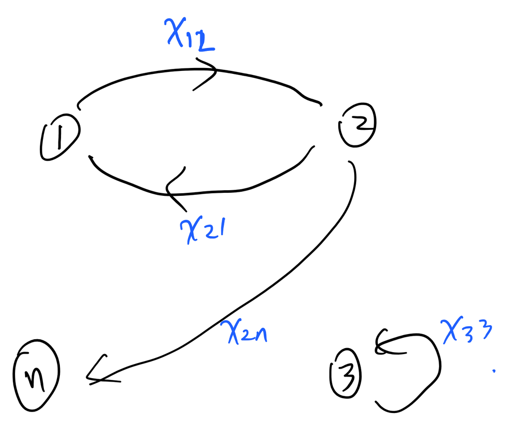
expression size

$VP_e \in VP$.

Quick Recap of DET & PERM.

Matrix $\longleftrightarrow$ Labeled Directed Graphs

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$



$G_n$.

$Perm_n = Perm(G_n)$

$Det_n = \det(G_n)$

Let G be a labeled directed graph

$$Perm(G) = \sum_{\text{cycle covers } \sigma} weight(\sigma)$$

$$Det(G) = \sum_{\text{cycle cover } \sigma} sgn(\sigma) \cdot weight(G)$$

Any cycle cover can be written as $\sigma = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_k$

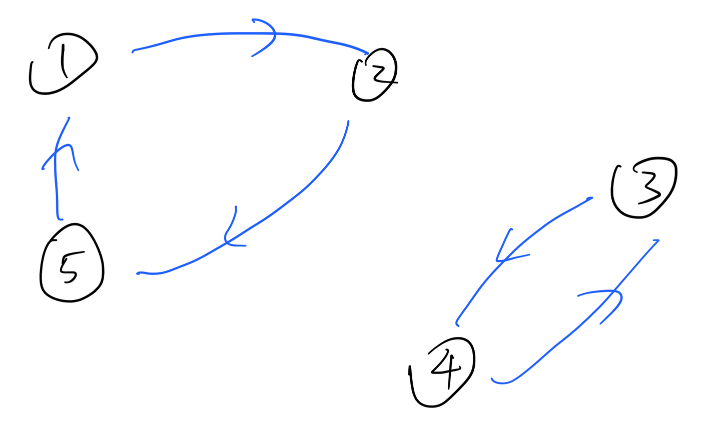
$Sgn(\sigma) = \prod (-1)^{|\sigma_i| - 1}$

Cycle Covers $\xleftrightarrow{\text{bijection}}$ * Permutations *

Assignment of the next vertex for every vertex. $\sim$ Bijection Map.

$\{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$
$i \longrightarrow \sigma(i)$

Ex|



$\{1, 2, \dots, 5\} \to \{1, 2, \dots, 5\}$

$1 \longmapsto 2 \;= \sigma(1)$
$2 \longmapsto 5$
$5 \longmapsto 1$
$3 \longmapsto 4$
$4 \longmapsto 3.$

Let $S_n = \{\text{Permutations on } \{1, \dots, n\}\}$

Let A $n \times n$ matrix

$$Perm_n A = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$$det_n A = \sum_{\sigma \in S_n} sgn(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

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Reduction in Alg. Comp. Theory.

Recall: If A poly reduces to B.

$B \in P$ (or NP) $\Longrightarrow$ $A \in P$ (or NP)

Def$^n$

We say $\{f_n\}$ is a p-projection of $\{g_n\}$

if $f_n(x_1, \dots, x_{m_n}) = g_{s_n}(L(x_1, \dots, x_{m_n}))$

where $L : k^{m_n} \longrightarrow k^{m'_{s_n}}$ where $m_n = \#$ variables of $f_n$
affine linear (and $S_n$ is poly bdd!) $m_{s_n} = \#$ --- $g_{s_n}$

$\hookrightarrow$ Linear map + constant

Ex 1| $y = mx + b$

$\mathbb{R} \to \mathbb{R}$

Ex 2| $\mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \mapsto (-2x+y+6,\ 3x+2y+3)$

$S_n$ is poly bounded.

FACT (HW)
If $\{f_n\}$ is a p-projection of $\{g_n\}$
then $\{g_n\} \in VP \longrightarrow \{f_n\} \in VP$.

TM
Non-Determinism in Complexity Thery.
At each step, branch along k different options
So after n steps, you have $k^n$ # branches

ACT
To compute "non-det" $\{f_n\}$, we allow to sum over $2^n$ variables
of another poly $\{g_n\} \in VP$.

Def$^n$ A sequence $\{f_n\}$ is p-definable if # variables of $f_n$ and $\deg(f_n)$ poly bounded, and either.
a) $\exists$ p-computable seq. $\{g_n\}$ with

$$f_n = \sum_{e_1, \dots, e_n \in \{0,1\}} g_n(e_1, \dots, e_n)\, x_1^{e_1} \cdots x_n^{e_n}$$

b). $\{f_n\}$ is a p-projection of a sequence $\{f_n\}$ as in (a).

Define
$VNP = \{ \{f_n\} \text{ p-definable} \}$.

We have
$VP \subset VNP$
Valient's conj.
$VP \neq VNP$.
$\downarrow$ $det_n$ $\quad$ $\downarrow$ $perm_n$.

Prop: $Perm_n \in VNP$.

$VNP = \{ \{f_n\} \mid$ coefficients of $f_n$ can