__HW__ 5/1/2019

PERFECT $\in P$

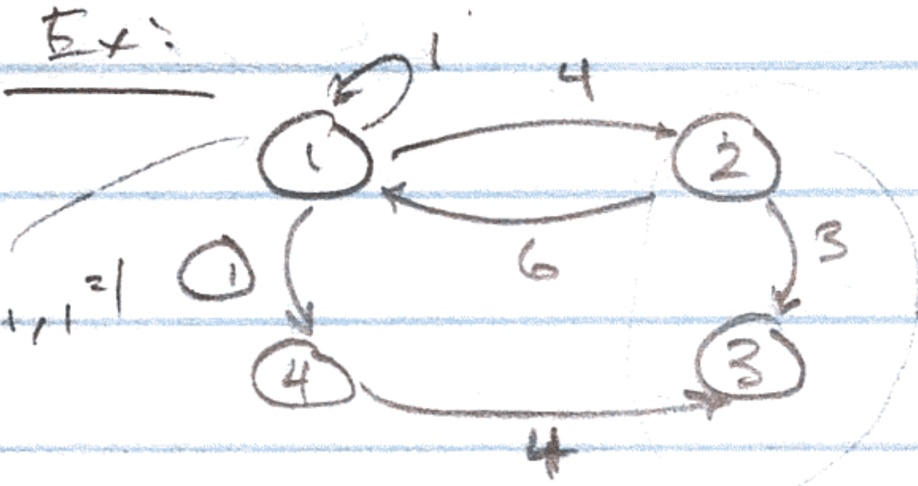
NP-complete is Counting Version

__Today:__ Determinants & permanents.

Approach via graphs

Let Graph G be directed graph w/ vertices $1, 2, \ldots, n$.

__Ex:__



$w_{1,1} = 1$

$w_{2,3} = 3$

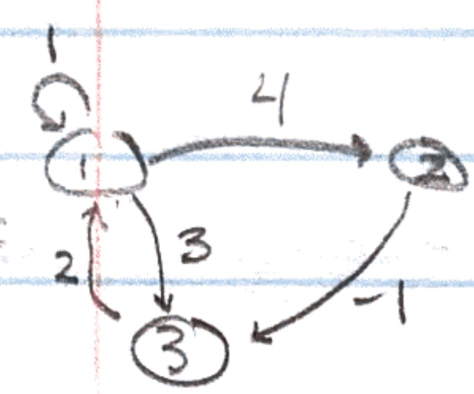
Let $w_{ij} = 0$ if $\nexists$ edge $i \to j$

__Def:__ Adjacency Matrix

$$A_G = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix}$$

__Ex:__

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 0 & -1 \\ 2 & 0 & 0 \end{pmatrix} \qquad G =$$

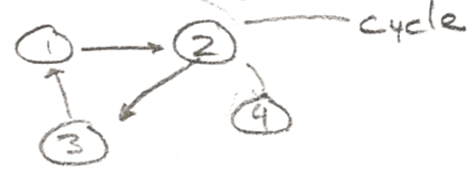


__Prop:__ $\exists$ 1:1 bijection $G \longrightarrow A_G$ & $G_A \longleftarrow A$.

__Def:__ Let G be labelled direct graph w/ vertices $1, \ldots, n$.

A cycle σ in G is path starting & ending @ same vertex & doesn't pass through the same vertex twice.

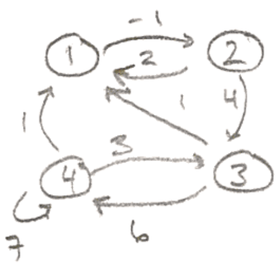
Ex:



Def:

Cycle cover = union of finitely many cycles such that every vertex in the graph is in precisely one cycle.

Ex: Let G be below:



Cycle Cover:

$\sigma_1: 1 \to 2 \to 3 \to 4 \to 1$

$\sigma_2: 1 \to 2 \to 1 \quad 3 \to 4 \to 3$

$\sigma_3: 1 \to 2 \to 3 \to 1 \quad 4 \to 4$

Cycle Notation

$(1\ 2\ 3\ 4)$

$(1\ 2)(3\ 4)$

$(1\ 2\ 3)(4)$

Weight:

$(-1)\cdot 4 \cdot 6 \cdot 1 = -24$

$(-1)\cdot 2 \cdot 3 \cdot 6 = -36$

$(-1)\cdot 4 \cdot 1 \cdot 7 = -28$

Def: If G is labelled directed graph:

$$\text{perm}(G) = \sum_{\sigma} \text{weight}(\sigma)$$

$$\det(G) = \sum_{\sigma} \text{sgn}(\sigma)\,\text{weight}(\sigma).$$

Ex:

$\text{perm}(G) = (-24) + (-36) + (-28) = -88$

$\det(G) = (-1)\cdot 24 + 1\cdot(-36) + 1\cdot(-28) = -40.$

If A is $n \times n$ matrix associated with graph G_A, then

$\text{perm}(A) := \text{perm}(G_A)$

$\det(A) := \det(G_A)$.

Def: Sign of a cycle cover:

$$\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_n$$

is

$$\text{sign}(\sigma) = \prod (-1)^{|\sigma_i| - 1}$$

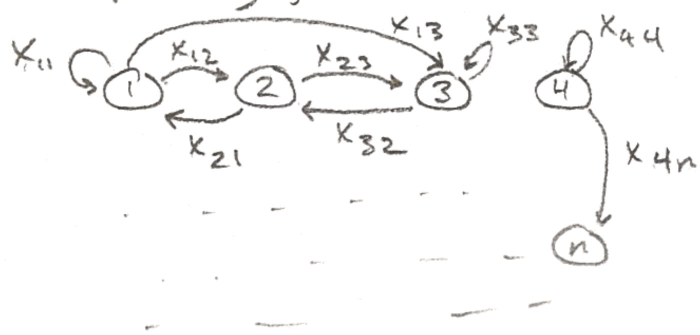
where $|\sigma_i|$ = # of vertices in cycle σ_i.

Det & Perm'n of poly.'s:

Consider the $n \times n$ matrix:

$$X_n = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & \vdots \\ x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix}$$

Corresponding graph is:



Def:

$\det_n := \det(X_n) = \det(G_n)$

$\text{perm}_n := \text{perm}(X_n) = \text{perm}(G_n)$.

$n=3$: $X_3 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ \vdots & & \vdots \\ x_{31} & \cdots & x_{33} \end{pmatrix}$ * You can draw graph.

$$\text{perm}_3 = x_{11}x_{22}x_{33} + x_{12}x_{21}x_{33} + \cdots + x_{13}x_{32}x_{21}$$

Prop: $\{\det_n\} \in VP$

Sketch: $\left.\begin{array}{l} \#\text{ of var.'s} = n^2 \\ \deg(\det_n) = n \end{array}\right\}$ poly bdd.

Complexity?

FACT: (Lin. alg). $\det(AB) = \det(A)\cdot\det(B)$.

Cor: Elementary row/column operations do not change det of matrix.

HW: G = bipartite graph

perf. matchings = perm (G).

For