Definition 1. A **Turing Machine** is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$$

Where Q is the finite set of states, Σ is the input alphabet, Γ is the tape alphabet ("$\llcorner$"$\in \Gamma$ and $\Sigma \subset \Gamma$), δ is the transition function ($\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$), q_0 is the initial state, and q_{accept} and q_{reject} are the accept and reject states.

1	0	1	0	...	1	␣	1	...

Definition 2. If $A \subset \Sigma *$ is a language, we say a Turing Machine (TM) M **recognizes** A if for all $w \in \Sigma *$

$$M \text{ accepts } w \Leftrightarrow w \in A.$$

Definition 3. A language A is **recognizable** if there exists a TM which recognizes it.
Example 1. *Let*

$$EVEN = \{string\ w_n \dots w_1\ in\ binary | n \in \mathbb{Z}, n = \Sigma w_i 2^i\ is\ even\}$$

Design a TM that recognizes $EVEN$.
Algorithm:

1. *If it reads a symbol that's not ␣ move to the right and repeat rule 1*
2. *If it reads ␣ move to the left and read that symbol*

Example 2. *Let*

$$NOTPRIME = \{n \in \mathbb{Z}, n > 0\ in\ binary\ which\ are\ not\ prime\} = \{1, 4, 6, 8, 9, \dots\}$$

Design a TM that recognizes $EVEN$.
Algorithm:

1. *Set $i = 2$*
2. *Divide the imput n by i, if it divides and $i \neq n$, accept, otherwise, $i = i + 2$ and repeat rule 2*

This TM recognizes $NOTPRIME$, but does not necessarily reject on $NOTPRIME^c$.
Algorithm:

1. *Set $i = 2$*
2. *Divide the imput n by i, if it divides and $i \neq n$, accept, if $i = n$, reject otherwise, $i = i + 2$ and repeat rule 2*

This TM recognizes $NOTPRIME$ and rejects on $NOTPRIME^c$.

Let $A \subset \Sigma *$:
Definition 4. A TM M **recognizes** A if for all $w \in \Sigma *$

$$M \text{ accepts } w \Leftrightarrow w \in A$$

Definition 5. A TM M **decides** A if for all $w \in \Sigma *$

$$M \text{ accepts } w \Leftrightarrow w \in A$$

$$M \text{ rejects } w \Leftrightarrow w \notin A$$

Definition 6. A TM with **multiple tapes** has multiple tapes and multiple heads

Theorem 1. *A TM with multiple tapes is equivalent to a TM with 1 tape*
Definition 7. A **non-deterministic** Turing Machine is similar to a TM, but at every step the machine can proceed in multiple ways simultaneously. i.e., "it can perform multiple branches at the same time."
Definition 8. A **power set** of a set S $\mathcal{P}(S)$ is set of subset of S. E.g.,

$$\mathcal{P}(\{0,1,2\}) = \{\{\},\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}$$

Example 3 (How an Non-Deterministic Turing Machine (NDTM) operates)**.**

γ_0	*0*	*1*	*0*	...	*1*	...

with the q_0 *at* γ_0 *and* $\delta(q_0,\gamma_0) \to \{(q_1,\gamma_1, L \text{ or } R),\ldots,(q_n,\gamma_n, L \text{ or } R)\}$. *Thus it goes from 1 to* n *branches in a single step. This can obviously grow exponentially.*
Example 4. *Let's design an efficient NDTM which recognizes* $NOTPRIME$.
Given input n

- *Let* $i = 2$.
- *For each possibility, divide* n *by corresponding integer. If it divides accept, otherwise* $i = i + 1$ *and repeat step 2.*