4/13/19 Week 2, Lecture 6

Today: Decidability. Are all languages decidable?

Set Theory

$\mathbb{N} = \{1, 2, 3, \dots\}$ natural numbers
$\mathbb{Z}$ = integers
$\mathbb{Q}$ = rationals
$\mathbb{R}$ = real #'s
$\mathbb{C}$ = complex #'s

def/ Two sets S_1 and S_2 are bijective if $\exists$ map $f: S_1 \to S_2$
s.t. ① f injective: for $x, y \in S$ with $f(x) = f(y)$, then $x = y$
② f surjective/onto if $\forall z \in S_2 \; \exists x \in S_1$ with $f(x) = z$

def/ a set S is countable if $\exists$ bijection $f: \mathbb{N} \to S$

rm/ $1 \mapsto f(1)$
$2 \mapsto f(2)$
$3 \mapsto f(3)$
This gives us a way to count or enumerate the elt's of S

prop/ $\mathbb{Z}$ is countable

pf/

$\mathbb{N}$: $\dots$ 7 5 3 1 2 4 6 $\dots$
$\mathbb{Z}$: $\dots$ -3 -2 -1 0 1 2 3 $\dots$

define $f: \mathbb{N} \to \mathbb{Z}$

$1 \mapsto 0$
$2 \mapsto 1$
$3 \mapsto -1$
$4 \mapsto 2$
$5 \mapsto -2$

$f(2i) = i$
$f(2i+1) = -i$
$f(1) = 0$

$\blacksquare$

prop/ $\mathbb{Q}$ is countable

pf/ Uses diagonalizing

$$\mathbb{Q} = \{ \tfrac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \}$$ not unique, ex. $\frac{2}{2} = \frac{1}{1}$

b \ a	1	2	3	4	5	6
1	1/1	2/1	3/1	4/1	5/1	
2	1/2	2/2	3/2	4/2		
3	1/3	2/3	3/3			
4	1/4	2/4				
5	1/5					
6						

$\Longrightarrow \mathbb{Q}_{>0} = \{ \tfrac{a}{b} > 0 \}$ is countable

$$\mathbb{Q} = \underbrace{\mathbb{Q}_{>0}}_{\text{countable}} \cup \underbrace{\mathbb{Q}_{<0}}_{\text{countable}} \cup \{0\}$$

$\Longrightarrow \mathbb{Q}$ countable ■

Fact/ (1) $\mathbb{R}$ is <u>not</u> countable
(2) power set $P(\{0,1\}^*)$ is not countable

$P(\{0,1\}^*)$ is similar to strings in $\{0,1\}$ of possibly ∞ length (bijective to the reals)

def/ Let $A \subset \{0,1\}^*$ language
(1) A is recognizable if $\exists$ TM M s.t. $\forall w \in \{0,1\}^*$
  M accepts $w \Leftrightarrow w \in A$
(2) A is decidable if $\exists$ TM M s.t. $\forall w \in \{0,1\}^*$
  M accepts $w \Leftrightarrow w \in A$
  M rejects $w \Leftrightarrow w \notin A$

th/ $\exists$ language $A \subset \{0,1\}^*$ that is not recognizable.

★

pf/ $\boxed{\text{Key Point: a TM is determined by finite data, i.e. given a TM } M \text{ we can encode } M \text{ as a binary string } [M] \in \{0,1\}^*.}$

Therefore set $S_1 := \{ A \subset \{0,1\}^* \mid \exists \text{ TM } M \text{ recognizes } A \}$

S_1 is countable b/c set $\{\text{Turing Machines } M\}$ is countable

(Homework: $\{0,1\}^*$ is countable)

Let S_2 = set of all languages
$= P(\{0,1\}^*)$

$S_1 \subset S_2$ and S_2 is uncountable $\Rightarrow \exists A \in S_2$ not in S_1
therefore S_1, S_2 not bijective $\not\exists$ ■

Let's define $A_{TM} = \{ ([M], w) \mid [M] \text{ is a TM in binary, } w \in \{0,1\}^* \text{ where } M \text{ accepts } w \}$

prop/ A_{TM} is recognizable

pf/ Design a TM N which recognizes A_{TM}
Algorithm: given input $([M], w)$
N simulates M on input w and accepts if M accepts. ■

rm/ M may not halt on input w, therefore N may also not halt.
So this doesn't decide A_{TM}.

th/ A_{TM} is not decidable (Halting Problem) pf) Suppose $\exists$ TM H which decides A_{TM}. Construct a new TM H' that does the opposite of what H does when H is run on its own input, i.e.

H' accepts $[M] \iff$ [redacted] M rejects ([redacted]) $[M]$
H' rejects $[M] \iff$ [redacted] M accepts ([redacted]) $[M]$

What happens if we run H' with input $[H']$?

~~H' accepts accept~~
H' accepts $[H'] \iff H'$ rejects $[H']$
H' rejects $[H'] \iff H'$ accepts $[H']$

This contradicts assumption that $\exists$ TM H that decides A_{TM}
$\implies A_{TM}$ is not decidable

ex) Liar's Paradox : "I am lying"
If I'm telling the truth, I'm lying
If I'm lying, I'm lying abt lying (telling the truth)