Math 480A: Algebraic Complexity Theory, Spring Quarter 2019
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Homework 3
Due: Monday, May 6

Problem 3.1. Determine all polynomials $f(x, y, z)$ in $k[x, y, z]$ which have complexity $C(f) \leq 3$.

Problem 3.2. Determine whether the following sequences of polynomials are p-computable.

(a) $f_n(x_1, \dots, x_n) = x_1^n + \cdots + x_n^n$

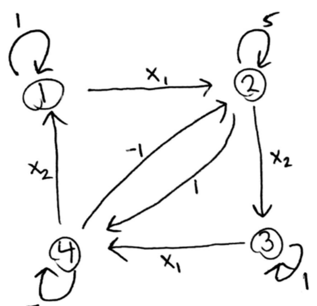
(b)
$$f_n(x_1, \dots, x_{n^2}) = \sum_{i=1}^{n^2} x_i^i$$

(c)
$$f_n(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i^i$$

(d)
$$f_n(x_1, \dots, x_n) = \sum_{\substack{I=(i_1,\dots,i_n) \\ |I|=n}} \frac{n!}{i_1! \cdots i_n!} x^I$$

Problem 3.3. Let $\{g_n\} \in \mathrm{VP}$ and let $\{f_n(x_1, \dots, x_{m_n})\}$ be a sequence of polynomials such that $\{m_n\}$ is polynomial bounded. Suppose that for each $n \geq 1$, there is an affine linear map $L_n \colon k^{m_n} \to k^{m'_n}$, where m'_n is the number of variables of g_n, such that
$$f_n(x_1, \dots, x_{m_n}) = g_n(L_n(x_1, \dots, x_{m_n})).$$
Show that $\{f_n\} \in \mathrm{VP}$.

Problem 3.4. Define the following directed labelled graph G



(a) Calculate $\mathrm{perm}(G)$.
(b) Calculate $\det(G)$.
(c) Determine the adjacency matrix of G.

Problem 3.5. Let G be an (undirected and unlabelled) bipartite graph with $2n$ vertices separated into two groups: $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$. Define an $n \times n$ matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ as follows: $a_{i,j} = 1$ if there exists an edge $v_i \to w_j$ in G; otherwise, $a_{i,j} = 0$. Show that $\mathrm{perm}(A)$ is the number of perfect matchings of G.