Math 427 Homework #7 Solutions

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Problem 7.1. Taylor 3.2.8.

Proof. Define function $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \sin(z)/z, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

We want to show g is entire. Since $\sin(z)$ is entire and z is analytic and not zero on $\mathbb{C} \setminus \{0\}$, it follows g is analytic on $\mathbb{C} \setminus \{0\}$, it suffices to show g is analytic around 0. Recall we have power series expansion for $\sin(z)$ around 0:

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \quad z \in \mathbb{C}.$$

Thus by definition of g,

$$g(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Let c_k denote the coefficient of the power series of $\sin(z)$, then $c_1 = (-1)^0/1! = 1$, then the power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}$ has value 1 at $z = 0$, which agrees with the value of g at 0. Therefore we conclude g has power series expansion

$$g(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}, \quad z \in \mathbb{C}.$$

Therefore it follows g is analytic around 0, hence g is entire. □

Problem 7.2. Taylor 3.2.9.

Proof. Since f is analytic on the disk $D_r(z_0)$, we have a power series expansion for f about z_0 with radius r (Theorem 3.2.5):

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad z \in D_r(z_0).$$

Since $f \neq 0$ on $D_r(z_0)$, it's not the case that $a_k = 0$ for all $k \geq 0$, thus there exists a minimal k such that $a_k \neq 0$. Then

$$f(z) = \sum_{n=k}^{\infty} a_k (z - z_0)^k = (z - z_0)^k \sum_{n=0}^{\infty} a_{k+n} (z - z_0)^n, \quad z \in D_r(z_0).$$

Define $g(z) := \sum_{n=0}^{\infty} a_{k+n}(z - z_0)^n$ for $z \in D_r(z_0)$, since by construction g has a power series expansion about z_0 with radius r, it is analytic on $D_r(z_0)$. Furthermore, by construction $a_k = g(z_0) \neq 0$. □

Lemma 0.1. *Recall given $z_0 \in \mathbb{C}$ we define $\lim_{z \to z_0} g(z) = \infty$ if for all $K > 0$ there exists $\delta > 0$ such that $|g(z)| > K$ whenever $|z - z_0| < \delta$.*

Let $g : \mathbb{C} \to \mathbb{C}$ be a function then $\lim_{z \to \infty} g(z) = \lim_{z \to 0} g(1/z)$.

Proof. We consider two cases:

1. $\lim_{z\to\infty} g(z) = \infty$. We want to show $\lim_{z\to 0} g(1/z) = \infty$ as well, in other words, for any $K > 0$ there exists $\delta > 0$ such that $|z| < \delta$ implies $|g(1/z)| > K$.

   Let $K > 0$ be given, since $\lim_{z\to\infty} g(z) = \infty$, there exists $M > 0$ such that $|z| > M$ implies $|g(z)| > K$. Define $\delta := 1/M > 0$, then if $|z| < \delta$, we have $|1/z| > 1/\delta = M$, then $|g(1/z)| > K$ as we wished.

2. Suppose $\lim_{z\to\infty} g(z) = L \in \mathbb{C}$. We want to show $\lim_{z\to 0} g(1/z) = L$ as well, in other words, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|z| < \delta$ implies $|g(1/z) - L| < \epsilon$.

   Let $\epsilon > 0$ be given, since $\lim_{z\to\infty} g(z) = L$, there exists $R > 0$ such that $|z| > R$ implies $|g(z) - L| < \epsilon$. Define $\delta := 1/R$, then if $|z| < \delta$, we have $|1/z| > 1/\delta = R$, then $|g(1/z) - L| < \epsilon$ as we wished.

□

Problem 7.3. Taylor 3.3.2.

Proof. By Lemma 0.1 it suffices to show $\lim_{z\to 0} f(1/z) = \infty$ if and only if $\lim_{z\to\infty} 1/f(z) = 0$.

1. Suppose $\lim_{z\to 0} f(1/z) = \infty$. Let $\epsilon > 0$ be given, we want $R > 0$ such that $|z| > R$ implies $|1/f(z)| < \epsilon$.

   Since $\lim_{z\to 0} f(1/z) = \infty$, there exists $\delta > 0$ such that $|z| < \delta$ implies then $|f(1/z)| > 1/\epsilon$. Define $R := 1/\delta$, then if $|z| > R$, we know $|1/z| < 1/R = \delta$, then $|f(1/(1/z))| = |f(z)| > 1/\epsilon$, then $|1/f(z)| < \epsilon$ as we wished.

2. Suppose $\lim_{z\to\infty} 1/f(z) = 0$. Let $K > 0$ be given, we want $\delta > 0$ such that $|z| < \delta$ implies $|f(1/z)| > K$.

   Since $\lim_{z\to\infty} 1/f(z) = 0$, there exists $R > 0$ such that $|z| > R$ implies $|1/f(z)| < 1/K$. Define $\delta := 1/R$, then if $|z| < \delta$, then $|1/z| > 1/\delta = R$, then $|1/f(1/z)| < 1/K$, then $|f(1/z)| > K$ as we wished.

□

Problem 7.4. Taylor 3.3.3.

Proof. Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, and $\lim_{z\to\infty} f(z) = \infty$, suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$, then the function $1/f$ is well-defined and analytic on the whole complex plane. By the previous problem we know $\lim_{z\to\infty} 1/f(z) = 0$, then we can find $R > 0$ such that $|1/f(z)| < 1$ whenever $|z| > R$. Since the closed disk $\overline{D}_R(0) = \{z \in \mathbb{C} : |z| \leq R\}$ is a compact and $1/f$ is continuous on it, the function $1/f$ attains a maximum value M at some $z_0 \in \overline{D}_R(0)$, therefore for all $z \in \mathbb{C}$, $|1/f(z)| \leq \max\{1, M\}$, the function $1/f$ is bounded on $\mathbb{C}$. By Liouville's Theorem the function $1/f$ must be constant, there's some $c \in \mathbb{C}$ such that $1/f(z) = c$ for all $z \in \mathbb{C}$. But this contradicts the assumption that $\lim_{z\to\infty} f(z) = \infty$ as there's no $z \in \mathbb{C}$ such that $|f(z)| > c$. □

Problem 7.5. Taylor 3.3.5.

Proof. Suppose we are given an entire function $f = u + iv$ such that u is bounded on $\mathbb{C}$. Consider $g(z) := e^{f(z)} = e^{u(z)}e^{iv(z)}$. Then $|g(z)| = \left|e^{u(z)}e^{iv(z)}\right| = \left|e^{u(z)}\right|$ is bounded on $\mathbb{C}$. Since f is entire and the exponential function is entire, by composition, g is entire, but g is also bounded, thus g is constant, i.e., there's some $c \in \mathbb{C}$ such that $g(z) = c$ for all $z \in \mathbb{C}$. Since $e^z \neq 0$ for all $z \in \mathbb{C}$, c cannot be zero, thus $f(z) = \log(c) \in \mathbb{C}$ for all $z \in \mathbb{C}$, f is constant. □

Problem 7.6. Taylor 3.3.6.

Proof. Let f be an entire nonconstant function, suppose for contradiction that $f(\mathbb{C})$ is not dense in $\mathbb{C}$, then there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $D_r(z_0) \subseteq \mathbb{C} \setminus f(\mathbb{C})$. Define $g(z) := 1/(f(z) - z_0)$, since $z_0 \notin f(\mathbb{C})$, the function g is well-defined and entire. We claim the function g must be bounded, it suffices to show there exists $\epsilon > 0$ such that $|f(z) - z_0| > \epsilon$ for all $z \in \mathbb{C}$. Put ϵ to be $r/2$ suffices, since for all $z \in \mathbb{C}$, $f(z) \notin D_r(z_0)$, which means $|f(z) - z_0| \geq r > r/2 = \epsilon$. Since the entire function g is bounded, it follows g is constant, and hence $f = z_0 + 1/g$ is also constant. □

Problem 7.7. Taylor 3.3.15.

Proof. Let p be a polynomial of degree n with real coefficients, then by fundamental theorm of algebra it factors into linear terms over $\mathbb{C}$:

$$p(z) = \lambda(z - z_1)(z - z_2)\cdots(z - z_n), \quad z_i, \lambda \in \mathbb{C}.$$

Problem 10 tells us that if $(z-r)$ is a factor of p, where $r \in \mathbb{C}\setminus\mathbb{R}$, then $(z-\overline{r})$ is also a factor of p. Therefore it suffices to show for $r \in \mathbb{C}\setminus\mathbb{R}$, $(z-r)(z-\overline{r})$ is a polynomial of degree at most 2 with real coefficients. Expand the product we get $(z-r)(z-\overline{r}) = z^2 - (r+\overline{r})zz + r\overline{r}$. Since $r + \overline{r} = Re(r)$ and $r\overline{r} = |r|^2$ are real, the product is a polynomial of degree 2 of real coefficients. ☐

Problem 7.8. Taylor 3.4.4.

Solution: Recall we have power series expansion of $\sin(z)$ around 0:

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \quad z \in \mathbb{C}.$$

Therefore $\sin(z) - z$ has power series expansion around 0:

$$\sin(z) - z = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z^3 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k-2}, \quad z \in \mathbb{C}.$$

Define $g(z) := \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k-2}$, then $g(0) = (-1)^1/3! \neq 0$, thus $\sin(z) - z$ has zero of order 3 at 0 with the factorization given above. ☐

Problem 7.9. Taylor 3.4.9.

Proof. Since f and g are analytic on U, and $z_0 \in U$, they have power series expansion around z_0, by Theorem 3.4.1 there exists $k, l \in \mathbb{N}$ and we may write

$$\begin{aligned} f(z) &= (z - z_0)^k \tilde{f}(z), \quad z \in D_{r_1}(z_0), \\ g(z) &= (z - z_0)^l \tilde{g}(z), \quad z \in D_{r_2}(z_0), \end{aligned}$$

where $\tilde{f}$ and $\tilde{g}$ are analytic on $D_{r_1}(z_0)$ and $D_{r_2}(z_0)$ respectively and $\tilde{f}(z_0), \tilde{g}(z_0) \neq 0$. Thus we know in particular that $\lim_{z\to z_0} \tilde{f}(z) = \tilde{f}(z_0) \neq 0$, $\lim_{z\to z_0} (\tilde{f})'(z) = (\tilde{f})'(z_0) \in \mathbb{C}$, $\lim_{z\to z_0} (\tilde{g})'(z) = (\tilde{g})'(z_0) \in \mathbb{C}$. Therefore,

$$\lim_{z\to z_0} \frac{f'(z)}{g'(z)} = \lim_{\substack{z\to z_0 \\ z\in D_{r_1}(z_0) \\ z\in D_{r_2}(z_0)}} \frac{f'(z)}{g'(z)} = \lim_{\substack{z\to z_0 \\ z\in D_{r_1}(z_0) \\ z\in D_{r_2}(z_0)}} \frac{[(z-z_0)^k \tilde{f}(z)]'}{[(z-z_0)^l \tilde{g}(z)]'}. = \lim_{z\to z_0} \frac{[(z-z_0)^k \tilde{f}(z)]'}{[(z-z_0)^l \tilde{g}(z)]'}.$$

Notice since $f(z_0) = g(z_0) = 0$, we must have $k, l \geq 1$. By product rule for complex differentiable functions,

$$\lim_{z\to z_0} \frac{f'(z)}{g'(z)} = \lim_{z\to z_0} \frac{k(z-z_0)^{k-1}\tilde{f}(z) + (z-z_0)^k (\tilde{f})'(z)}{l(z-z_0)^{l-1}\tilde{g}(z) + (z-z_0)^l (\tilde{g})'(z)}.$$

We consider the following three cases:

1. $k = l$. In this case

$$\lim_{z\to z_0} \frac{f'(z)}{g'(z)} = \lim_{z\to z_0} \frac{k\tilde{f}(z) + (z-z_0)(\tilde{f})'(z)}{k\tilde{g}(z) + (z-z_0)(\tilde{g})'(z)}$$

   Using the theorem on sum, product and quotient of converging limits, we conclude

$$\lim_{z\to z_0} \frac{f'(z)}{g'(z)} = \frac{k\tilde{f}(z_0) + 0}{k\tilde{g}(z_0) + 0} = \frac{\tilde{f}(z_0)}{\tilde{g}(z_0)}.$$

2. $k < l$. Then
$$\lim_{z\to z_0}\frac{f'(z)}{g'(z)} = \lim_{z\to z_0}\frac{k\tilde{f}(z)+(z-z_0)(\tilde{f})'(z)}{l(z-z_0)^{l-k}\tilde{g}(z)+(z-z_0)^{l-k+1}(\tilde{g})'(z)}.$$
Observe the limit of the numerator exists and nonzero,
$$\lim_{z\to z_0} k\tilde{f}(z)+(z-z_0)(\tilde{f})'(z) = k\tilde{f}(z_0)\in\mathbb{C}\setminus\{0\}.$$
However the limit of the denominator is
$$\lim_{z\to z_0} l(z-z_0)^{l-k}\tilde{g}(z)+(z-z_0)^{l-k+1}(\tilde{g})'(z) = 0+0=0.$$
Thus we conclude the $\lim_{z\to z_0} f'(z)/g'(z)$ is unbounded, in which case we denote the limit by ∞.

3. $l < k$. Then
$$\lim_{z\to z_0}\frac{f'(z)}{g'(z)} = \lim_{z\to z_0}\frac{k(z-z_0)^{k-l}\tilde{f}(z)+(z-z_0)^{k-l+1}(\tilde{f})'(z)}{l\tilde{g}(z)+(z-z_0)(\tilde{g})'(z)}.$$
Using the theorem on sum, product and quotient of converging limits, we conclude,
$$\lim_{z\to z_0} k(z-z_0)^{k-l}\tilde{f}(z)+(z-z_0)^{k-l+1}(\tilde{f})'(z) = 0+0=0,$$
and
$$\lim_{z\to z_0} l\tilde{g}(z)+(z-z_0)(\tilde{g})'(z) = l\tilde{g}(z_0)\neq 0.$$
Therefore, $\lim_{z\to z_0} f'(z)/g'(z) = 0$.

□

Problem 7.10. Exercises Taylor 3.4.12–16.

Solution:

1. Observe the function $z - z^3$ is entire, and it is only zero when $z = 0$ or $z = \pm 1$, thus f is analytic on $\mathbb{C}\setminus\{0,1,-1\}$. If there were isolated singularities of f, they may only occur at $0, 1$ and -1. First we look at 0, near the origin the function f has factorization $f(z) = z^{-1}g(z)$ where $g(z) = 1/[(z+1)(z-1)]$, which is analytic around 0, thus f has a pole of order 1 at 0. Secondly we look at 1, similarly we have factorization $f(z) = (z-1)^{-1}g(z)$ where $g(z) = 1/[z(z+1)]$, which is analytic around 1, thus f has a pole of order 1 at 1. Finally by the same reasoning, f has a pole of order 1 at -1 as well.

2. By power series expansion of $\sin(z)$, we have
$$\sin(1/z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{-(2k+1)}, \quad z\in\mathbb{C}\setminus\{0\}.$$
Since there are infinitely many terms in the power series expansion of $\sin(1/z)$ about 0 where z has negative exponent. Thus $\sin(1/z)$ has an essential singularity at 0. Away from 0 however, the function $1/z$ is analytic, hence by composition $\sin(1/z)$ is analytic on $\mathbb{C}\setminus\{0\}$, hence 0 is the only isolated singularity of $\sin(1/z)$.

3. Since the numerator of f is $e^z - 1 - z$, which is entire; the denominator of f is z^2, which is also entire. It follows the isolated singularity of f may only occur at where denominator vanishes, which happens only if $z = 0$. Recall we have power series expansion of e^z about 0:
$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}, \quad z\in\mathbb{C},$$

then we have

$$f(z) = \frac{\left(\sum_{n=0}^{\infty} z^n/n!\right) - 1 - z}{z^2} = \sum_{n=2}^{\infty} \frac{z^{n-2}}{n!}, \quad z \in \mathbb{C} \setminus \{0\}.$$

If we define $f(0)$ to be $1/2$, the function f has power series expansion $\sum_{n=2}^{\infty} \frac{z^{n-2}}{n!}$ around 0, hence we conclude f has a removable singularity at 0.

4. Observe $f(z) = \frac{z+1-e^z}{z(e^z-1)}$, and both the numerator and the denominator are entire and the denominator is only zero when $z = 2\pi i n$ for $n \in \mathbb{Z}$. Thus the isolated singularity of f may only occur at those points. By change of variable we have power series expansion of $e^{z-2\pi i n}$ around $2\pi i n$:

$$e^{z-2\pi i n} = \sum_{k=0}^{\infty} \frac{(z-2\pi i n)^k}{k!}, \quad z \in \mathbb{C}.$$

Observe since $e^{2\pi i n} = 1$, the left hand side equals e^z, thus we have power series expansion of e^z around $2\pi i n$:

$$e^z = \sum_{k=0}^{\infty} \frac{(z-2\pi i n)^k}{k!}, \quad z \in \mathbb{C}.$$

Therefore we have power series expansion of the numerator around $2\pi i n$:

$$z + 1 - e^z = 2\pi i n - \sum_{k=2}^{\infty} \frac{(z-2\pi i n)^k}{k!}, \quad z \in \mathbb{C}.$$

If $n = 0$, it can be factored as

$$z + 1 - e^z = z^2\Big(-\sum_{k=2}^{\infty} \frac{z^{k-2}}{k!}\Big) =: z^2 g(z).$$

Thus the numerator is entire and has a zero of order 2 at 0, it does not have a zero at $2\pi i n$ for $n \neq 0$. Consider the denominator $z(e^z - 1)$, observe the function z is entire and is zero when $n = 0$, it's nonzero when $n \neq 0$. Furthermore the power series expansion of $e^z - 1$ around $2\pi i n$ has factorization

$$e^z - 1 = (z - 2\pi i n) \sum_{k=1}^{\infty} \frac{(z-2\pi i n)^{k-1}}{k!}.$$

Therefore we conclude the denominator is entire and has a zero of order 2 at 0, and has a zero of order 1 at $2\pi i n$ when $n \neq 0$. We conclude using Example 3.4.10 that f has a removable singularity at 0 and a pole at $2\pi i n$ for $n \in \mathbb{Z}, n \neq 0$. Recall to remove a removable singularity, we define the value of the function to be the constant term in its power series expansion around that singularity. Since f has power series expansion around 0:

$$f(z) = \frac{z^2(-\sum_{k=2}^{\infty} z^{k-2}/k!)}{z^2(\sum_{k=1}^{\infty} z^{k-1}/k!)} = \frac{-\sum_{k=2}^{\infty} z^{k-2}/k!}{\sum_{k=1}^{\infty} z^{k-1}/k!},$$

the constant term is the constant term of the numerator $(-1/2)$ divided by the constant term of the denominator 1, which is $-1/2$.

5. Recall the principal branch of the logarithm $\log(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$. Since the denominator $(1-z)^2$ is entire and is zero only when $z = 1$, it follows f is analytic on $\mathbb{C} \setminus ((-\infty, 0] \cup \{1\})$.

   Observe for all $z_0 \in (-\infty, 0]$, and all open ball containing z_0, the open ball also contains points in $(-\infty, 0]$, this tells us none of points in $(-\infty, 0]$ is an isolated singularity of f, we only need to consider the point $z_0 = 1$.

Recall we have power series expansion of $\log(1+z)$ around 0:

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \quad |z| < 1,$$

then by a change of variable, we have a power series expansion of $\log(z)$ around 1:

$$\log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n, \quad |z-1| < 1.$$

We can factor this expansion into:

$$\log(z) = (z-1)\Big(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^{n-1}\Big) =: (z-1)g(z), \quad |z-1| < 1.$$

Then g is analytic around 0 and $g(1) \neq 0$, thus $\log(z)$ has a zero of order 1 at 1. Since the denominator $(z-1)^2$ has a zero of order 2 at 1, it follows from Example 3.4.10 that f has a pole of order 1 at 1.

$\square$