# Math 427 Homework #5 Solutions

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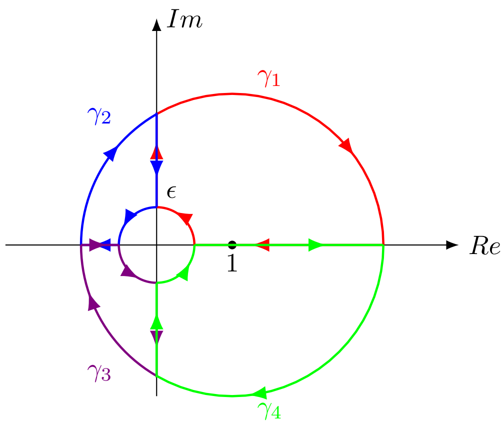
**Problem 5.1. Taylor 2.6.4.**

**Solution:** Suppose 0 is in the interior of $\gamma$, then there exists $\epsilon > 0$ such that the closed ball $\overline{B}_\epsilon(0)$ is contained entirely in the interior of $\gamma$. Split the regions into the following:




Then by additivity of countour integral we have

$$\int_\gamma \frac{dz}{z} = \sum_{i=1}^{4} \int_{\gamma_i} \frac{dz}{z} + \int_{|z|=\epsilon} \frac{dz}{z}.$$

Since each $\gamma_i$ is contained in a convex open set where $1/z$ is analytic on, each $\int_{\gamma_i} dz/z$ is zero. Therefore,

$$\int_\gamma \frac{dz}{z} = \int_{|z|=\epsilon} \frac{dz}{z} = \int_0^{2\pi} \frac{\epsilon i e^{it}}{\epsilon e^{it}} dt = 2\pi i.$$

Suppose on the other hand that $\gamma$ is the circle centered at $z_0$ with radius $r$, and 0 is not contained in the interior of $\gamma$, then there exists $\epsilon > 0$ such that the circle centered at $z_0$ with radius $r + \epsilon$ does not contain 0 as well. Since $D_{r+\epsilon}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r + \epsilon\}$ is open convex and does not contain 0, hence the function $1/z$ is analytic on it. It follows by Cauchy's Integral Theorem that $\int_\gamma dz/z = 0$. □

**Problem 5.2. Taylor 2.6.7.**

*Proof.* Let $\gamma_1$ and $\gamma_2$ be two paths in $U$ both begin at $a$ and end at $b$, where $a, b \in U$. Then the path $\gamma_1 - \gamma_2$, the path construted by $\gamma_1$ followed by the reverse of $\gamma_2$ is a closed path in $U$ starting and ending at $a$. Since $f$ is analytic on $U$ it follows by Cauchy's Integral Theorem that the integral $\int_{\gamma_1 - \gamma_2} f dz$ is zero. Since countour integral is additive in paths, we know by Theorem 2.4.6

$$\int_{\gamma_1 - \gamma_2} f dz = \int_{\gamma_1} f dz - \int_{\gamma_2} f dz = 0.$$

Therefore it follows

$$\int_{\gamma_1} f dz = \int_{\gamma_2} f dz,$$

as we wished. □

**Problem 5.3. Taylor 2.6.10.**

**Solution:**

$$\begin{aligned}\mathrm{Ind}_\gamma(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{dw}{w - z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ine^{int}dt}{z_0 + e^{int} - z_0} \\ &= \frac{1}{2\pi i}\int_0^{2\pi} indt = \frac{2\pi in}{2\pi i} = n.\end{aligned}$$

□

**Problem 5.4. Taylor 2.6.13.**

**Solution:**

1. By partial fraction we can decompose the integrand $1/(z^2 - 1)$ into:

$$\frac{1}{z^2 - 1} = \frac{1}{2}\Big(\frac{1}{z-1} - \frac{1}{z+1}\Big).$$

   Then by linearity the integral becomes:

$$\frac{1}{2}\int_{|z-1|=1} \frac{dz}{z-1} - \frac{1}{2}\int_{|z-1|=1} \frac{dz}{z+1}.$$

   For the first integral, we may apply a change of variable $w := z-1$ and observe it equals $(1/2)\int_{|w|=1} dw/w$, which by Problem 5.1 Taylor 2.6.4 we know equals $\pi i$. We apply the change of variable $w := z + 1$ in the second integral and turn it into $(1/2)\int_{|w-2|=1} dw/w$. Observe it integrates $1/w$ around the circle of radius 1 centered at 2 (which does not have 0 in its interior), again by Problem 5.1 we conclude the integral is 0. Therefore the answer to this part is $\pi i$.

2. Use the same partial fraction decomposition we turn the integral into

$$\frac{1}{2}\int_{|z+1|=1} \frac{dz}{z-1} - \frac{1}{2}\int_{|z+1|=1} \frac{dz}{z+1}.$$

   Apply change of variable $w := z - 1$ in the first integral and $w := z + 1$ in the second integral, we turn the above into

$$\frac{1}{2}\int_{|w+2|=1} \frac{dw}{w} - \frac{1}{2}\int_{|w|=1} \frac{dw}{w}.$$

   The first integral integrates $1/w$ around a circle of radius 1 centered at $-2$, which does not include 0 in its interior. By Problem 5.1 we see its zero. The second integral by Problem 5.1 is $2\pi i$. Therefore the answer to this part is $-\pi i$.

□

**Lemma 0.1.** *Let $f, g$ be continuous functions defined on an open set $U$, suppose $f$ is differentiable on $U$. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path, such that $\gamma([a, b]) \subset U$, then*

$$\int_\gamma f'(z)g(z)dz = \Big[(f\circ\gamma)(t)(g\circ\gamma)(t)\Big]_a^b - \int_\gamma f(z)g'(z)dz.$$

*Proof.* By definition of the contour integral,

$$\int_\gamma f'(z)g(z)dz = \int_a^b f'(\gamma(t))g(\gamma(t))\gamma'(t)dt = \int_a^b (f\circ\gamma)'(t)(g\circ\gamma)(t)dt.$$

By the integration by parts of complex-valued (i.e. vector valued) functions, we can rewrite the above integral as:

$$\Big[(f\circ\gamma)(t)(g\circ\gamma(t))\Big]_a^b - \int_a^b (f\circ\gamma)(t)(g\circ\gamma)'(t)dt.$$

Then by definition of contour integral, we have

$$\Big[(f\circ\gamma)(t)(g\circ\gamma(t))\Big]_a^b - \int_a^b f(\gamma(t))g'(\gamma(t))\gamma'(t)dt = \Big[(f\circ\gamma)(t)(g\circ\gamma(t))\Big]_a^b - \int_\gamma f(z)g'(z)dz,$$

as we wished. □

**Problem 5.5.** Let $\gamma$ be the counterclockwise parameterization of the unit circle centered at the origin. Compute the integral

$$\int_\gamma \frac{e^z}{z^n}dz$$

for every integer $n$.

**Solution:**

1. Suppose $n \leq 0$, then $n = -m$ for some integer $m \geq 0$. Then the integrand $e^z/z^n = e^z z^m$ is analytic on the entire complex plane. By Cauchy's Integral Theorem the integral is zero.

2. Suppose $n \geq 1$, we induction on $n$ to prove the integral equals $2\pi i/(n-1)!$. Suppose $n = 1$, then by Cauchy's Integral Formula,

$$\int_\gamma \frac{e^z}{z}dz = (2\pi i)\,\mathrm{Ind}_\gamma(0)e^0 = 2\pi i = \frac{2\pi i}{0!}.$$

Suppose $n > 1$, integration by parts (Lemma 0.1) tells us that

$$\int_\gamma \frac{e^z}{z^n}dz = \int_\gamma \Big(\frac{z^{-n+1}}{-n+1}\Big)' e^z dz = \Big[\frac{e^{i(-n+1)t}}{-n+1}\cdot e^{e^{it}}\Big]_0^{2\pi} - \int_\gamma \frac{z^{-n+1}}{-n+1}e^z dz.$$

By inductive hypothesis

$$\int_\gamma \frac{e^z}{z^{n-1}}dz = \frac{2\pi i}{(n-2)!},$$

therefore

$$\int_\gamma \frac{e^z}{z^n}dz = \Big[\frac{e^{i(-n+1)t}}{-n+1}\cdot e^{e^{it}}\Big]_0^{2\pi} + \frac{1}{n-1}\frac{2\pi i}{(n-2)!} = \frac{2\pi i}{(n-1)!}.$$

This concludes the induction.

□

**Problem 5.6.** Let $\gamma$ be the clockwise parameterization of the circle of radius 2 centered at $1+i$. Compute

$$\int_\gamma \frac{dz}{z^4-1}.$$

*Proof.* By partial fraction, we have

$$\frac{1}{z^4-1} = \frac{1}{4}\Big(\frac{-1}{z+1} + \frac{1}{z-1} - \frac{i}{z+i} + \frac{i}{z-i}\Big).$$

Therefore the integral becomes

$$-\frac{1}{4}\int_\gamma \frac{dz}{z+1} + \frac{1}{4}\int_\gamma \frac{dz}{z-1} - \frac{i}{4}\int_\gamma \frac{dz}{z+i} + \frac{i}{4}\int_\gamma \frac{dz}{z-i}.$$

Make change of variable $w = z+1$, $w = z-1$, $w = z+i$ and $w = z-i$ in the four integrals respectively, we get

$$-\frac{1}{4}\int_{|w-(2+i)|=2} \frac{dw}{w} + \frac{1}{4}\int_{|w-i|=2} \frac{dw}{w} - \frac{i}{4}\int_{|w-(1+2i)|=2} \frac{dw}{w} + \frac{i}{4}\int_{|w-1|=2} \frac{dw}{w}.$$

Recall the path $\gamma$ were parametrized to travel clockwise, hence the four integrals above are integrated around a circle clockwise once. We may compute each of them using Problem 5.1. That is, if the circle we are integrating over has 0 in its interior, then the result of the integral is $-2\pi i$, otherwise the integral is 0. The first and third does not, and the second and fourth does. Therefore the integral is

$$\int_\gamma \frac{dz}{z^4-1} = \frac{-2\pi i}{4} + \frac{i(-2\pi i)}{4} = -\frac{\pi i}{2} + \frac{\pi}{2}.$$

□

**Problem 5.7. Taylor 2.7.1.**

*Proof.* Let $\{U_\alpha\}_{\alpha\in I}$ be a collection of connected sets in $\mathbb{C}$, suppose a common $p$ is contained in the intersection $\cap_{\alpha\in I}U_\alpha$. For contraction suppose $\cup_{\alpha\in I}U_\alpha$ is separated: there exists open sets $A, B \subset \mathbb{C}$ such that $\cup_{\alpha\in I}U_\alpha \subset A\cup B$, $\Big(\cup_{\alpha\in I} U_\alpha\Big)\cap A \neq \varnothing$, $\Big(\cup_{\alpha\in I} U_\alpha\Big)\cap B \neq \varnothing$ and $A\cap B = \varnothing$.

Since $p \in \cup_{\alpha\in I}U_\alpha \subset A\cup B$, and $A$ is disjoint from $B$ either $p\in A$ or $p \in B$. Without loss of generality suppose $p\in A$, since $p\in \cap_{\alpha\in I}U_\alpha$, we must have $U_\alpha \cap A \neq \varnothing$ for all $\alpha\in I$. Since for each $\alpha \in I$, we have $U_\alpha \subset A\cup B$, $U_\alpha\cap A\neq\varnothing$ and $A\cap B$. Since $U_\alpha$ is connected, it cannot have a separation, then it must be the case that $U_\alpha \cap B = \varnothing$. Therefore,

$$\Big(\bigcup_{\alpha\in I} U_\alpha\Big)\cap B = \bigcup_{\alpha\in I} U_\alpha\cap B = \varnothing,$$

a contradiction to the fact that $A\cup B$ is a separation. □

**Problem 5.8. Taylor 2.7.8.**

**Solution:** Suppose the path travel the figure eight in the direction below: We have three connected components

$$\begin{aligned} A &= \{z\in\mathbb{C} : |z-i|>1 \text{ and } |z+i|>1\} \\ B &= \{z\in\mathbb{C} : |z-i|<1\} \\ C &= \{z\in\mathbb{C} : |z+i|<1\} \end{aligned}$$

Since $A$ is unbounded, $\mathrm{Ind}_\gamma(z) = 0$ for all $z\in A$. Crossing $\gamma$ from $A$ to $B$ at $2i$ increases the index by 1, thus $\mathrm{Ind}_\gamma(z) = 1$ for all $z\in B$. Similarly when we cross $\gamma$ from $A$ to $C$ at $-2i$ decreases the index by 1, thus $\mathrm{Ind}_\gamma(z) = -1$ for all $z \in C$. □

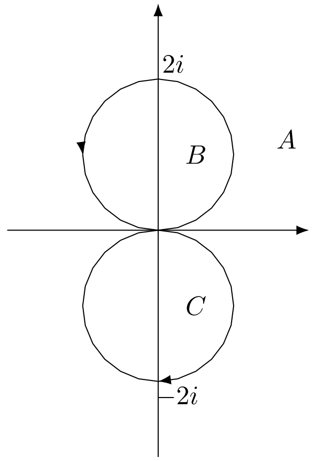



Figure 1: Problem 5.8

**Problem 5.9. Taylor 2.7.9.**

**Solution:** Apply Theorem 2.7.8. □

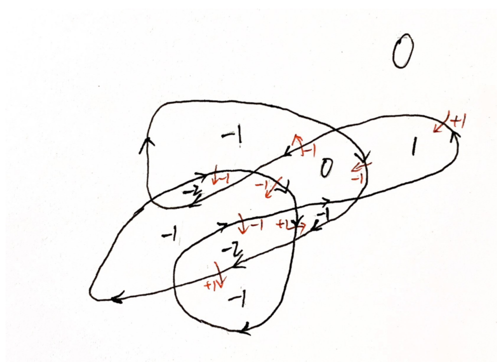


Figure 2: Problem 5.9

**Problem 5.10. Taylor 2.7.10.**

*Proof.* By definition,

$$\operatorname{Ind}_{\gamma_1+\gamma_2}(z) = \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2}\frac{d\zeta}{\zeta - z}, \quad \operatorname{Ind}_{-\gamma_1}(z) = \frac{1}{2\pi i}\int_{-\gamma_1}\frac{d\zeta}{\zeta - z}.$$

Since $z$ is not on either paths, the function $1/(\zeta - z)$ is continuous as $\zeta$ runs through $\gamma_1$ and $\gamma_2$, by Theorem 2.4.6, we may rewrite the integrals above as:

$$\operatorname{Ind}_{\gamma_1+\gamma_2}(z) = \frac{1}{2\pi i}\Big(\int_{\gamma_1}\frac{d\zeta}{\zeta - z} + \int_{\gamma_2}\frac{d\zeta}{\zeta - z}\Big), \quad \operatorname{Ind}_{-\gamma_1}(z) = -\frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta}{\zeta - z}.$$

Therefore it follows,

$$\operatorname{Ind}_{\gamma_1+\gamma_2}(z) = \operatorname{Ind}_{\gamma_1}(z) + \operatorname{Ind}_{\gamma_2}(z), \quad \operatorname{Ind}_{-\gamma_1}(z) = -\operatorname{Ind}_{\gamma_1}(z).$$

□