Math 427 Homework #2 Solutions

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Problem 2.1. Taylor 1.4.1.

Solution: To find the polar form of a complex number z, we first need to find the modulus of z, then find the argument of z, which is the angle the number makes with the positive real axis. Notice the choice of the argument is not unique, they may differ up to an integer multiple of 2π.

1. The modulus is 1, the argument is $\pi + 2k\pi$, where $k \in \mathbb{Z}$, thus $-1 = e^{i(\pi+2k\pi)}$ $(k \in \mathbb{Z})$.
2. The modulus is 1, the argument is $\pi/2 + 2k\pi$, where $k \in \mathbb{Z}$, hence $i = e^{i(\pi/2+2k\pi)}$ $(k \in \mathbb{Z})$.
3. The modulus is 1, the argument is $3\pi/2 + 2k\pi$, where $k \in \mathbb{Z}$, hence $-i = e^{i(3\pi/2+2k\pi)}$ $(k \in \mathbb{Z})$.
4. The modulus is $\sqrt{1 + \sqrt{3}^2} = 2$, the argument is $\pi/3 + 2k\pi$, where $k \in \mathbb{Z}$, hence $1 + \sqrt{3}i = 2e^{i(\pi/3+2k\pi)}$ $(k \in \mathbb{Z})$.
5. The modulus is $\sqrt{5^2 + (-5)^2} = 5\sqrt{2}$, the argument is $-\pi/4 + 2k\pi$, where $k \in \mathbb{Z}$, hence $5 - 5i = 5\sqrt{2}e^{i(-\pi/4+2k\pi)}$ $(k \in \mathbb{Z})$.

Pick any $k \in \mathbb{Z}$ gives a correct solution. □

Problem 2.2. Taylor 1.4.3.

Solution: Recall from the textbook that the n-th power of $e^{\pi i/8}$ is $e^{(\pi i n)/8}$. Also recall $e^{2\pi i} = 1$. Therefore, given any integer n, we can uniquely write $n = 16k + r$ for some integer k and $r = n \bmod 16$, then

$$(e^{\pi i/8})^n = e^{(\pi i(16k+r))/8} = e^{2k\pi i}e^{\pi i r/8} = e^{\pi i r/8}.$$

Because there are 16 distinct integers modulo 16, there are precisely 16 distinct powers of $e^{\pi i/8}$, which are $e^{\pi i r/8}$ for $r = 0, \ldots, 15$. □

Problem 2.3. Taylor 1.4.6.

Proof. Since z is on the unit circle, we know $|z| = 1$.

("if") Suppose the argument of z is a rational multiple of 2π, then we can write the argument of z as $(p/q)2\pi + 2k\pi$, where $k \in \mathbb{Z}$ and $p/q \in \mathbb{Q}$. Therefore, the n-th power of z is $|z|^n e^{i[(p/q)2n\pi+2nk\pi]} = e^{i[(p/q)2n\pi]}$. We can uniquely write n in the form $n = 2ql + r$, for some integer l and $r = n \bmod 2q$. Then substitute $n = 2ql + r$ into the n-th power of z, we get

$$\begin{aligned} e^{i[(p/q)2\pi(2ql+r)]} &= e^{i[(p/q)4ql\pi+(p/q)2r\pi]} = e^{i(p/q)4ql\pi}e^{i(p/q)2r\pi} = (e^{i2\pi})^{2pl}e^{i(p/q)2r\pi} \\ &= e^{i(p/q)2r\pi} = z^r. \end{aligned}$$

This shows that n-th powers of z is equal to $(n \bmod 2q)$-th power of z. Since there are finitely many distinct integers modulo $2q$, there are only finitely many distinct powers of z.

("only if") Suppose there are only finitely many distinct powers of z, then there must exist integers n and m with $n > m$ such that $z^n = z^m$. But this means that $z^{n-m} = 1$. Since z is on the unit circle, we can write $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$, then we would have $e^{i(n-m)\theta} = 1$. This implies $(n - m)\theta = 2k\pi$ for some positive integer k, or in other words that $\theta = \frac{2k\pi}{n-m}$ is a rational multiplie of 2π.

□

Problem 2.4. Taylor 1.4.10.

Solution:

1. The angle $-i$ makes with the positive real axis in the complex plane is $-\pi/2 + 2k\pi$ radians, since we are interested in the argument that lies in $I = (-\pi, \pi]$, we may take $k = 0$, and the argument is $-\pi/2$.
2. The angle $-i$ makes with the positive real axis in the complex plane is $-\pi/2 + 2k\pi$ radians, since we are interested in the argument that lies in $I = (0, 2\pi]$, we may take $k = 1$, and the argument is $-\pi/2 + 2\pi = 3\pi/2$.
3. The number $z = 1$ lies on the positive real axis, hence it has argument $2k\pi$ for some $k \in \mathbb{Z}$. Since we are only interested in the argument that lies in $[3\pi/2, 7\pi/2)$, we may take $k = 1$, and the argument is $2\pi \in [3\pi/2, 7\pi/2)$.

□

Problem 2.5. Taylor 1.4.11.

Solution: The modulus of $1-i$ is $\sqrt{1^2 + (-1)^2} = \sqrt{2}$, the argument of $1-i$ in the principal branch $(-\pi, \pi]$ is $-\pi/4$, hence the in principal branch of the log function,

$$\log(1 - i) = \log|1 - i| + i \arg_{(-\pi,\pi]}(1 - i) = \log(\sqrt{2}) + i(-\pi/4).$$

□

Problem 2.6. Taylor 1.4.16.

Solution: Since we are using the principal branch of the log function to define the square root in this problem, the cut line for $\sqrt{z}$ is the half-line of negative real axis. We want to find cut line(s) for z such that when z crosses those line(s), $1 - z^2$ crosses the negative real axis, then those cut lines will be the set of discontinuities for the function $\sqrt{1 - z^2}$. We will find those lines by working backwards. When $1 - z^2$ crosses the negative real axis counterclockwise, $-z^2$ crosses the line $(-\infty, -1)$ counterclockwise, hence z^2 crosses the line $(1, \infty)$ counterclockwise. For that to happen z must crosses either the line $(1, \infty)$ or $(-\infty, -1)$ counterclockwise. Therefore the set of discontinuities for $\sqrt{1 - z^2}$ is $\{x + i0 : x \in \mathbb{R}, |x| > 1\}$. □

Problem 2.7. Taylor 2.1.4.

Proof. Let $w = x + iy$ be a point in A, we want to show w is the center of an open disc that is entirely contained in A. By definition we know $Re(w) = x > 0$, hence we may pick $r := x/2 > 0$ and consider the disc $D_r(w)$. We claim the disc $D_r(w)$ is contained in A. Let $\zeta = a + ib \in D_r(w)$ be given, by definition we know $|\zeta - w| < r$, by triangle inequality, we know

$$|a - x| \leq |(a - x) + i(b - y)| = |\zeta - w| < r = x/2.$$

In particular, we know

$$-x/2 < a - x.$$

Adding x on both sides we get inequality

$$x/2 < a.$$

Since $x/2 > 0$, it follows $a = Re(\zeta) > 0$, therefore $\zeta \in A$ by definition. This proves $D_r(w) \subset A$, hence proves the claim that A is open. □

Problem 2.8. Taylor 2.1.5.

Solution:

(a) Open.

(b) Neither. It's not open since every open disc centered at $1/2$ contains a point out side of the set. It's not closed since every open disc of 0 contains a point of the set.

(c) Closed.

□

Problem 2.9. Taylor 2.1.6.

Solution: The interior is the set
$$\{z \in \mathbb{C} : 1 < |z| < 2\}.$$
The closure is the set
$$\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}.$$
The boundary is the set
$$\{z \in \mathbb{C} : |z| = 1\}.$$

□

Problem 2.10. Taylor 2.1.9.

Proof.

1. The function $Re(z)$ is continuous on $\mathbb{C}$. Let $z_0 = x_0 + iy_0 \in \mathbb{C}$ be picked, we want to show $\lim_{z \to z_0} Re(z) = Re(z_0)$. That is: for any $\epsilon > 0$, there exists $\delta > 0$ such that $|Re(z) - Re(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.
   Let $\epsilon > 0$ be picked, set $\delta = \epsilon$, and let $z = x + iy \in \mathbb{C}$ with $|z - z_0| < \delta = \epsilon$ be given. By the triangle inequality, we know
   $$|Re(z) - Re(z_0)| = |x - x_0| \leq |(x - x_0) + i(y - y_0)| = |z - z_0| < \epsilon.$$
   Which is what we wanted.
2. One could modify the above argument to show that $Im(z)$ is continuous. We provide however an alternative argument here. Since the identity function $\mathrm{id}(z) = z = Re(z) + iIm(z)$ is continuous on $\mathbb{C}$ and the function $Re(z)$ is continuous on $\mathbb{C}$, it follows the function $iIm(z)$ is continuous on $\mathbb{C}$. Since the nonzero constant function i is continuous on $\mathbb{C}$, it follows $Im(z) = (iIm(z))/i$ is continuous on $\mathbb{C}$.
3. Since the functions $Re(z)$, $Im(z)$ and the constant function i are continuous on $\mathbb{C}$, it follows the function $\overline{z} = Re(z) - iIm(z)$ is continuous on $\mathbb{C}$.

□

Problem 2.11. Taylor 2.1.14.

Proof.

1. Observe we can rewrite the set as
   $$\{z \in U : |f(z)| < r\} = f^{-1}(D_r(0)),$$
   which is the preimage of a continuous function f of an open set, which is open by Theorem 2.1.13 of Taylor.

2. Let S denote the set $\{z \in \mathbb{C} : Re(z) < r\}$, we first show S is open. Let $z = x + iy \in S$ be given, by definition we know $x < r$, we claim the open disk $D_{(r-x)/2}(z)$ is contained in S: let $w = a + ib \in D_{(r-x)/2}(z)$ be given, by definition $|w - z| < (r - x)/2$, then by triangle inequality,

$$|a - x| \leq |(a - x) + i(b - y)| = |w - z| < (r - x)/2.$$

In particular we get

$$a - x < r/2 - x/2,$$

by adding x to both sides, we get inequality

$$Re(w) = a < r/2 + x/2 < r/2 + r/2 = r,$$

which by definition tells us that $w \in S$. This shows S is open.

Observe we can write the set of interest as

$$\{z \in U : Re(f(z)) < r\} = f^{-1}(S),$$

which is the preimage of a continuous function of an open set, which is open by Theorem 2.1.13 of Taylor.

□