Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 9
Due: Friday, May 28

Problem 9.1. Let p be a prime. Let $\rho = e^{2\pi i/p}$ be a primitive pth root of unity.

(a) Prove that $\mathbb{Q} \subseteq \mathbb{Q}(\rho)$ is a Galois field extension and that the Galois group $\mathrm{Gal}(\mathbb{Q}(\rho)/\mathbb{Q})$ is isomorphic to the multiplicate group $(\mathbb{Z}/p)^\times$ of units in $\mathbb{Z}/p$.
(b) Let $L = \mathbb{Q}(\rho)$. Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq L' \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

Problem 9.2. Let N and H be finite groups. Denote by $\mathrm{Aut}(N)$ the group of automorphisms of N. Let $\varphi\colon H \to \mathrm{Aut}(N)$ be a group homomorphism. Define the following group operation on the set $N \times H$ via

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2).$$

(a) Show that $N \times H$ is a group with the operation $\bullet$. We call this the *semi-direct product of N and H via φ* and denote it by $N \rtimes_\varphi H$.
(b) Show that H and N are naturally subgroups of $N \rtimes_\varphi H$.
(c) Show that N is a normal subgroup.
(d) Show that the dihedral group

$$D_{2n} = \{\sigma, \tau \mid \sigma^n = \tau^2 = \mathrm{id}, \sigma\tau = \tau\sigma^{-1}\}$$

is isomorphic to the semi-direct product $\mathbb{Z}/n \rtimes_\varphi \mathbb{Z}/2$ where $\varphi\colon \mathbb{Z}/2 = \{0, 1\} \to \mathrm{Aut}(\mathbb{Z}/n)$ is the group homomorphism where $\varphi(0)\colon \mathbb{Z}/n \to \mathbb{Z}/n$ is the identity and $\varphi(1)\colon \mathbb{Z}/n \to \mathbb{Z}/n$ sends $x \mapsto -x$.

Problem 9.3. Prove that the Galois group of the splitting field of $x^p - 2$ over $\mathbb{Q}$ for a prime p is isomorphic to the semi-direct product

$$\mathbb{Z}/p \rtimes_\varphi (\mathbb{Z}/p)^\times$$

where $\varphi\colon (\mathbb{Z}/p)^\times \to \mathrm{Aut}(\mathbb{Z}/p)$ is the group homomorphism such that $\varphi(a)$ is the automorphism of $\mathbb{Z}/p$ defined by multiplication by a.

Problem 9.4. Recall that a finite group G is said to be *solvable* if there exists a chain of subgroups

$$1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{k-1} \subseteq H_k = G$$

such that for each $i = 0, \ldots, k-1$, the subgroup $H_i \subseteq H_{i+1}$ is normal and $H_{i+1}/H_i \cong \mathbb{Z}/p_i$ for some prime p_i.

(a) Show that any subgroup H of a solvable group is also solvable.
(b) If H is a normal subgroup of a finite group G, show that G is solvable if and only if both H and G/H are solvable.
(c) Show that every finite abelian group is solvable.
(d) Use (c) to conclude that in the definition of solvable, it is equivalent to require that the quotients H_{i+1}/H_i be abelian.

Problem 9.5.

(a) Show that the dihedral group D_8 is solvable.
(b) Show that the alternating group A_4 is solvable.
(c) Show that the symmetric group S_4 is solvable.