Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 7
Due: Friday, May 14

Problem 7.1. Show that any field extension $K \subseteq L$ of degree 2 is normal.

Problem 7.2. Prove that every element of a finite field can be written as the sum of two squares.

Problem 7.3. Prove that there exists an inclusion of fields $\mathbb{F}_{p^a} \subseteq \mathbb{F}_{p^b}$ if and only if $a|b$.

Problem 7.4. Let p be a prime and $q = p^n$. Consider the map

$$\begin{aligned} \sigma \colon \mathbb{F}_q &\to \mathbb{F}_q \\ x &\mapsto x^p. \end{aligned}$$

(a) Show that σ is a well-defined homomorphism of fields.
(b) Show that σ is the identity on the subfield $\mathbb{F}_p \subset \mathbb{F}_q$.
(c) Show that $\sigma \colon \mathbb{F}_q \to \mathbb{F}_q$ is an isomorphism.
(d) Show that the set of elements fixed by σ is precisely $\mathbb{F}_p$; in other words, show that

$$\mathbb{F}_p \cong \{x \in \mathbb{F}_q \mid \sigma(x) = x\}.$$

If $K \subset L$ is a field extension, an *automorphism of* K *over* L is a field isomorphism $\sigma \colon L \to L$ that is the identity on K, i.e. for all $x \in K$, $\sigma(x) = x$. Let $\mathrm{Gal}(L/K)$ be the set of all automorphisms σ of K over L.

Problem 7.5. Let $K \subset L$ be a field extension.

(a) Show that $\mathrm{Gal}(L/K)$ is a group under composition.
(b) Let $\alpha \in L$ be a root of a polynomial $f(x) \in K[x]$. If σ is an automorphism of L over K, show that $\sigma(\alpha)$ is also a root of $f(x)$.

Problem 7.6.

(a) Determine $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$.
(b) Determine $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$.
(c) Determine $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i)/\mathbb{Q})$.
(d) Determine $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$.