Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 6
Due: Friday, May 7

Problem 6.1. Determine the splitting fields $\mathbb{Q} \subset K$ of the following polynomials defined over $\mathbb{Q}$ and compute the degree $|K : \mathbb{Q}|$.

(a) $f(x) = x^4 + 1$.
(b) $f(x) = x^3 - 3x + 2 = 0$

Hint: In Homework Problem 1.5, you solved for the roots of (b).

Problem 6.2.

(a) Show that $\mathbb{Q}(\sqrt{2}, i)$ is the splitting field of $x^2 - 2\sqrt{2}x + 3$ over $\mathbb{Q}(\sqrt{2})$.
(b) Find a polynomial $f(x) \in \mathbb{Q}[x]$ whose splitting field is $\mathbb{Q}(\sqrt[3]{2}, i, \sqrt{3})$.

Problem 6.3. Let K be a field and $L = K(\alpha)$ be a simple field extension of K. If L is normal over K, show that L is the splitting field of the minimal polynomial of α.

Problem 6.4.

(a) Count the number of monic irreducible polynomials over $\mathbb{F}_3$ of degree 2, 3 and 4.
(b) For each $d = 2, 3, 4$, explicitly exhibit a monic irreducible polynomial $f \in \mathbb{F}_3$ of degree d.

Problem 6.5. For each of the following field extensions, determine (a) whether it is normal and (b) whether it is separable.

(a) $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{-5})$.
(b) $\mathbb{Q}(i) \subseteq \mathbb{Q}(\sqrt[3]{2}, i)$.
(c) $\mathbb{F}_p \subseteq \mathbb{F}_{p^n}$ where p is a prime.
(d) $\mathbb{F}_p(x^p) \subseteq \mathbb{F}_p(x)$ where p is a prime.

Clarification: You may use the following facts (to be proven in lecture):

- *If the characteristic of K is zero, then any field extension $K \subseteq L$ is separable.*
- *If the characteristic of K is p and every element of K has a pth root, then any field extension $K \subseteq L$ is separable.*