Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 5
Due: Friday, April 30

Problem 5.1.

(a) Show that any angle can be bisected using only a ruler and a compass.
(b) Show that the angle 90° can be trisected using only a ruler and a compass.
(c) *Extra credit (5 points):* Show that the angle θ can be trisected using only a ruler and a compass if and only if the polynomial

$$4x^3 - 3x - \cos\theta$$

is reducible over $\mathbb{Q}(\cos\theta)$.

Problem 5.2.

(a) Show that a square can be doubled (i.e. it is possible to construct a new square with twice of the area of the original square) using only a ruler and a compass.
(b) Given a line segment AB, construct a square whose side lengths are $|AB|$ using only a ruler and a compass.

Problem 5.3. Given a circle, construct a regular hexagon that inscribes the circle using only a ruler and compass.

Problem 5.4. Let η be a primitive 9th root of unity.

(a) What is the minimal polynomial for η?
(b) Express η^{-1} as a $\mathbb{Q}$-linear combination of $1, \eta, \eta^2, \dots, \eta^5$.

Problem 5.5. Recall that the *splitting field of a polynomial* $f(x) \in \mathbb{Q}[x]$ is the smallest subfield $K \subset \mathbb{C}$ containing $\mathbb{Q}$ and all the roots of $f(x)$. Determine the splitting fields $\mathbb{Q} \subseteq K$ of the following polynomials defined over $\mathbb{Q}$ and compute the degree $|K : \mathbb{Q}|$.

(a) $f(x) = x^3 - 2$.
(b) $f(x) = x^4 - 3$.
(c) $f(x) = x^9 - 1$.

Problem 5.6. Show that the multiplicative group $\mathbb{F}_{11}^{\times}$ of non-zero elements is isomorphic to $\mathbb{Z}/10\mathbb{Z}$.