Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 4
Due: Friday, April 23

Problem 4.1. Use Lagrange's method to solve the quartic $x^4 + x + 3/4 = 0$.

Problem 4.2. Recall that $\mathbb{F}_2$ denotes the field $\mathbb{Z}/2$ with 2 elements.
(a) How many irreducible polynomials $f(x) \in \mathbb{F}_2[x]$ are there of degree 3?
(b) What about degree 4?
(c) Show explicitly that $\mathbb{F}_2[x]/(x^3 + x + 1)$ and $\mathbb{F}_2[x]/(x^3 + x^2 + 1)$ are isomorphic.

Problem 4.3. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \mathbb{C}$. Find all intermediate field extensions

$$\mathbb{Q} \subset K \subset L.$$

Problem 4.4. Determine (with proof) the degrees of the following field extensions, and write down an explicit basis for each:
(a) $\mathbb{Q} \subset \mathbb{Q}(1 + \sqrt[3]{2} + \sqrt[3]{4})$
(b) $\mathbb{Q} \subset \mathbb{Q}(e^{2\pi i/p})$ for a prime p
(c) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{10 + 4\sqrt{6}}, \sqrt{6})$

Problem 4.5.
(a) Find the minimal polynomial of $\alpha = 1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.
(b) Find the minimal polynomial of $\sqrt{10 + 4\sqrt{6}}$ over $\mathbb{Q}$.
(c) Find the minimal polynomial of $\sqrt{10 + 4\sqrt{6}}$ over $\mathbb{Q}(\sqrt{6})$.
(d) Find the minimal polynomial of $\sqrt{\pi} + \sqrt{3}$ over $\mathbb{Q}(\pi)$.

Problem 4.6. Prove that a field extension $K \subset L$ of prime degree is simple.