Math 404A: Introduction to Modern Algebra (Spring 2021)
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Homework 2
Due: Friday, April 9

You do not need to do Problem 2.3—it will be on next week's homework.

Problem 2.1. For each of the following actions of a group G on a set X, determine the orbit $Gx = \{g \cdot x \mid g \in G\}$ and stabilizer $G_x = \{g \in G \mid g \cdot x = x\}$ for each element $x \in X$:

(a) $G = S_3$ acting on $X = S_3$ via conjugation: for $g \in G$ and $x \in X$, the action is defined by $g \cdot x = gxg^{-1}$.
(b) $G = \{(1), (12), (345), (354), (12)(345), (12)(354)\} \subset S_6$ acting on $X = \{1, 2, 3, 4, 5, 6\}$ via permutation.

To check your answer, recall that for any $x \in X$*, then* $|G| = |G_x||Gx|$.

Problem 2.2.

(a) Express $x_1^4 + x_2^4 + x_3^4 + x_4^4$ as a polynomial in terms of the elementary symmetric functions s_1, s_2, s_3, s_4.
(b) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote the complex roots of the polynomial

$$x^4 + x^3 + 2x^2 + 3x + 5.$$

Determine the number $\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4$.

Hint: Do not actually solve for the roots α_i explicitly!

Problem 2.3. (Moved to HW3) Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial with roots $\alpha_1, \ldots, \alpha_n$. The *discriminant* of $f(x)$ is defined as

$$\Delta = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

(a) Prove that this is a symmetric function.
(b) If $f(x) = x^3 + a_2x + a_3$, express the discriminant Δ in terms of coefficients a_2, a_3.

Problem 2.4. Let $f \in k[\alpha_1, \ldots, \alpha_n]$ be any polynomial. Let $f_1, \ldots, f_k$ be the orbit of f under the action of S_n. (Note that one of the f_i is equal to f.)

(a) Show that $f_1 + \cdots + f_k$ is symmetric.
(b) If $s(x_1, \ldots, x_k)$ is any symmetric polynomial in $x_1, \ldots, x_k$, show that $s(f_1, \ldots, f_k)$ is a symmetric polynomial in $\alpha_1, \ldots, \alpha_n$.

Challenge Problem 2.5. (5 points) Let

$$f_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$$
$$f_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)$$
$$f_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)$$

(a) Express $f_1 + f_2 + f_3$ as a polynomial in terms of the elementary symmetric functions $s_1, \ldots, s_4$.
(b) Express $f_1f_2 + f_1f_3 + f_2f_3$ as a polynomial in $s_1, \ldots, s_4$.
(c) Express $f_1f_2f_3$ as a polynomial in $s_1, \ldots, s_4$.

Problem 2.6.

(1) Verify that Lagrange's method to solve the quartic yields the correct solutions to $x^4 + a_0 = 0$.
(2) Use Lagrange's method to solve $x^4 + x + 1 = 0$.

Challenge Problem 2.7. (10 points) Try to use a similar method to Lagrange's solution of the quartic to derive a solution to the cubic equation.

Hint: Search for a polynomial $f(\alpha_1, \alpha_2, \alpha_3)$ whose orbit under S_3 consists of only 2 elements (or in other words whose stabilizer subgroup has 3 elements).

Challenge Problem 2.8. (10 points) In 1540, Da Coi posed the following challenge to Cardan: Divide 10 into three parts such that they shall be in continued proportion and the product of the first two is 6.[1]

Use Lagrange's method to solve the above classical problem.

[1] Cardan states it as:

> Exemplum. Fac ex 10 tres partes proportionales, ex quarum ductu primæin secundam, producantur 6. Hanc proponebat Ioannes Colla, & dicebat solui non posse, ego uero dicebam, eam posse solui, modum tame ignorabam, donec Ferrarius eum inuenit." *Ars Magna* cap. XXXIV, qvæstio V; 1545 ed., fol. 73, v.

See pg.467 *History of Mathematics, Band 2* by David Eugene Smith for an account of the history of this problem.