Midterm 1

Modern Algebra (Math 403)
Instructor: Jarod Alper
Winter 2018
January 29, 2018 **Name:**

Read all of the following information before starting the exam:

- You may not consult any outside sources (calculator, phone, computer, textbook, notes, other students, ...) to assist in answering the exam problems. All of the work will be your own!
- Write clearly!! You need to write your solutions carefully and clearly in order to convince me that your solution is correct. Partial credit will be awarded.
- Good luck!

Problem	Points
1 (25 points)	_______
2 (25 points)	_______
3 (25 points)	_______
4 (25 points)	_______
Total (100 points)	

Problem 1. Write down a composition series for the dihedral group of order 12

$$D_6 = \langle r, s \mid r^6 = s^2 = 1, rs = sr^5 \rangle.$$

Solution: Consider the sequence of subgroups

$$0 \subset \langle r^2 \rangle \subset \langle r \rangle \subset D_6.$$

Observe that

- $D_6/\langle r \rangle \cong \mathbb{Z}/2$;
- $\langle r \rangle/\langle r^2 \rangle \cong \mathbb{Z}/2$; and
- $\langle r^2 \rangle \cong \mathbb{Z}/3$.

Each factor above is a cyclic group of prime order which we know from lecture is a simple group. It follows that the above sequence is a composition series.

Problem 2. Let $f(x) = 6x^5 + 2x^3 - x + 1$ and $g(x) = x^2 + 1$ be polynomials in $\mathbb{Q}[x]$. Find polynomials $q, r \in \mathbb{Q}[x]$ such that $f = qg + r$ with $\deg r < \deg g$ or $r = 0$.

Solution: We perform the division algorithm. Noticing that $6x^3 g$ and f have the same leading term of $6x^5$, we compute that $f - 6x^3 g = -4x^3 - x + 1$. Since $-4xg$ and $f - 6x^3 g$ have the same leading term, we compute that $f - (6x^3 - 4x)g = 3x + 1$. Thus, if we set $q = 6x^3 - 4x$ and $r = 3x + 1$, we have that $f = qg + r$.

Problem 3. Prove that there is a ring isomorphism

$$\mathbb{C}[x]/(x^2+1) \cong \mathbb{C} \times \mathbb{C}.$$

Solution: Observe that $x^2 + 1$ factors over $\mathbb{C}$ as $x^2 + 1 = (x + i)(x - i)$. If we define the ideals $I = (x + i)$ and $J = (x - i)$, we see that $I + J = (1)$ (since $\frac{1}{2i}((x+i)-(x-i)) = 1$). By the Chinese Remainder Theorem for Rings (Problem 3.5–Judson 16.6.40), we have an isomorphism of rings

$$\mathbb{C}[x]/(I \cap J) \to \mathbb{C}[x]/I \times \mathbb{C}[x]/J.$$

We also have that the ring homomorphism $\phi\colon \mathbb{C}[x]/(x + i) \to \mathbb{C}$ defined by $\phi(f) = f(-i)$ is an isomorphism. Likewise, $\mathbb{C}[x]/(x - i) \cong \mathbb{C}$. Finally, we claim that $I \cap J = (x^2+1)$. Clearly, $(x^2+1) \subset I \cap J$. On other hand, suppose $f \in I \cap J$. Then both $x+i$ and $x-i$ divides f which in turn implies that $x^2+1 = (x-i)(x+i)$ divides f. Therefore $I \cap J \subset (x^2 + 1)$. Combining the above observations, we obtain the desired isomorphism.

Alternatively, we can define a map

$$\varphi\colon \mathbb{C}[x] \to \mathbb{C} \times \mathbb{C}, \quad f \mapsto (f(i), f(-i)).$$

One needs to check that: (1) φ is a ring homomorphism, (2) φ is surjective and (3) $\ker(\varphi) = (x^2 + 1)$. It is easy to check that (1) holds (details not included here as we've already given a complete proof above). For (2), set $f_1 = \frac{1}{-2i}(x - i)$ and $f_2 = \frac{1}{2i}(x + i)$. Clearly, $\varphi(f_1) = (0, 1)$ and $\varphi(f_2) = (1, 0)$. It follows that f is surjective since for any $(a_1, a_2) \in \mathbb{C} \times \mathbb{C}$, we have that $f(a_1 f_1 + a_2 f_2) = (a_1, a_2)$. For (3), observe that $f \in \ker(\varphi)$ if and only if $f(i) = f(-i) = 0$, that is, both i and $-i$ are roots. The latter conditions holds if and only both $x+i$ and $x-i$ divides f, which is equivalent to $x^2 - 1$ dividing f. Therefore, $\ker(\varphi) = (x^2 + 1)$ and we may appeal to the first isomorphism theorem to conclude that $\mathbb{C}[x]/(x^2 + 1) \cong \mathbb{C} \times \mathbb{C}$.

Problem 4.

(a) Let R be a commutative ring. Show that any maximal ideal of R is also prime.

(b) Give an example of a commutative ring R and an ideal $\mathfrak{p} \subset R$ which is prime but not maximal.

Solution: For (a), we could appeal to the following two facts from lecture. If $I \subset R$ is an ideal, then I is prime if and only if R/I is an integral domain, and I is maximal if and only if R/I is a field. Since fields are integral domains, we see that maximal ideals are prime.

Alternatively, we could argue directly. Let $\mathfrak{m} \subset R$ be a maximal ideal and suppose $xy \in \mathfrak{m}$. Let us suppose that both x and y are not in $\mathfrak{m}$ and we'll try to get a contradiction. Since $\mathfrak{m}$ is maximal and $x, y \notin \mathfrak{m}$, we have that $\mathfrak{m} + (x) = \mathfrak{m} + (y) = R$. Therefore, we can write $1 = z_1 + r_1 x$ and $1 = z_2 + r_2 y$ with $z_1, z_2 \in \mathfrak{m}$ and $r_1, r_2 \in R$. By taking the product of these expressions, we have that

$$1 = 1 \cdot 1 = (z_1 + r_1 x)(z_2 + r_2 y) = z_1 z_2 + r_1 x z_2 + z_1 r_2 y + r_1 r_2 xy,$$

but since $z_1, z_2, xy \in \mathfrak{m}$, the right hand expression is in the maximal ideal $\mathfrak{m}$. This shows that $1 \in \mathfrak{m}$, which is a contradiction. Thus $x \in \mathfrak{m}$ or $y \in \mathfrak{m}$, and $\mathfrak{m}$ is prime.

For (b), take $R = \mathbb{Z}$ and the ideal $\mathfrak{p} = (0)$. Clearly $\mathfrak{p}$ is prime but it is not maximal (for instance, $(0) \subsetneq (2) \subsetneq \mathbb{Z}$).