Math 403B: Introduction to Modern Algebra, Winter Quarter 2018
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Homework 7
Due: Friday, March 9

Problem 7.1. Factor the polynomial $f = -6x^3 + 6x^2y^2 + 6x^3y - 3xy + 3xy^2 \in \mathbb{Z}[x, y]$ as a product of irreducible elements.

Problem 7.2.

(a) Factor 13 and 17 over $\mathbb{Z}[i]$.
(b) Write $221 = 13 \cdot 17$ as a sum of two squares in two different ways.

Hint: Using the factorization from (a), write 221 in two distinct ways as $(a + bi)(a - bi)$ with $a, b \in \mathbb{Z}$.

Problem 7.3. Let $\omega = e^{2\pi i/3}$ and let $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Show that $\mathbb{Z}[\omega]$ is a Euclidean Domain.

Problem 7.4. Let $a, b \in \mathbb{Z}$. Show that the greatest common divisor of a and b in $\mathbb{Z}$ is equal to the greatest common divisor of a and b in $\mathbb{Z}[i]$.

Problem 7.5. Find a generator of the ideal of $\mathbb{Z}[\omega]$ generated by 5 and $3 - \omega$.

Problem 7.6. Let $p \neq 3$ be a prime integer. Prove the following:

(a) The polynomial $x^2 + x + 1$ has a root in $\mathbb{Z}/p$ if and only if $p \equiv 1 \mod 3$.
(b) The ideal $(p) \subset \mathbb{Z}[\omega]$ is maximal if and only if $p \equiv 2 \mod 3$.
(c) The element $p \in \mathbb{Z}[\omega]$ factors in $\mathbb{Z}[\omega]$ if and only if p can be written as $p = a^2 + ab + b^2$ for integers a and b.

Problem 7.7. Show that $3 \in \mathbb{Z}[\sqrt{-5}]$ is an irreducible element that is not prime.

Problem 7.8. For which integers n does the circle defined by $x^2 + y^2 = n$ contain a point with integer coordinates?