Math 403B: Introduction to Modern Algebra, Winter Quarter 2018
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Homework 6
Due: Wednesday, February 28

Part 1: You are responsible for this material on the 2nd midterm

Problem 6.1.

(a) Show that the Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain (and therefore a PID and UFD).

(b) Factor the elements 5 and $6 + 8i$ in $\mathbb{Z}[i]$ as a product of irreducible elements.

Problem 6.2. Prove that $k[x, y]$ and $\mathbb{Z}[x]$ are not PIDs.

Problem 6.3. Let $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ be the subring of $\mathbb{C}$.

(a) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

Hint: Show that 6 has two distinct factorizations into irreducible elements. You need to prove that the elements appearing in your factorization are indeed irreducible.

(b) Recall that in lecture we proved that any PID is a UFD. It follows from (a) that $\mathbb{Z}[\sqrt{-5}]$ is not a PID. Write down an explicit ideal $I \subset \mathbb{Z}[\sqrt{-5}]$ which is not principal.

Problem 6.4. Judson 18.3.8

Problem 6.5. Judson 18.3.9

Problem 6.6. Judson 18.3.10

Problem 6.7. Judson 18.3.11

Problem 6.8. Judson 18.3.12

Problem 6.9. Let R be a UFD. Show that an element $x \in R$ is prime if and only if $x \in R$ is irreducible.

Part 2: You are *NOT* responsible for this material on the 2nd midterm

Problem 6.10. Let R be a commutative ring. We say that R satisfies the *ascending chain condition* if for every increasing sequence of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$, there exists an integer N such that $I_N = I_{N+1} = I_{N+2} = \cdots$. Additionally, we say that an ideal $I \subset R$ is *finitely generated* if there exists elements $a_1, \ldots, a_n$ such that $I = (a_1, \ldots, a_n)$. Here by definition $(a_1, \ldots, a_n) \subset R$ is the ideal of elements in R that can be written as sums $r_1a_1 + \cdots + r_na_n$ for some $r_1, \ldots, r_n \in R$.

Prove that R satisfies the ascending chain condition if and only if every ideal $I \subset R$ is finitely generated. Such a ring R is called *Noetherian*, after Emmy Noether.

Problem 6.11 (Hilbert Basis Theorem)**.** Follow the below steps to prove the following remarkable theorem due to David Hilbert: if R is Noetherian, then $R[x]$ is Noetherian.

(a) Let $I \subset R[t]$ be an ideal. For each positive integer d, define $I_d \subset R$ to be the set of elements $r \in R$ such that there a polynomial $rx^d + a_{d-1}x^{d-1} + \cdots + a_0 \in I$ for some $a_0, \ldots, a_{d-1} \in R$. Show that $I_d \subset R$ is an ideal.
(b) Show that $I_0 \subset I_1 \subset I_2 \subset \cdots$.
(c) Use that R is Noetherian to conclude that there is some integer N such that $I_N = I_{N+1} = I_{N+2} = \cdots$.
(d) For each $j = 0, \ldots, N$, choose a finite set of generators $g_{j,1}, \ldots, g_{j,n_j} \in R$ such that $I_j = (g_{j,1}, \ldots, g_{j,n_j})$. For each $j = 0, \ldots, N$ and $k = 1, \ldots, n_j$, choose $f_{j,k} \in I$ such that $f_{j,k} = g_{j,k}x^j +$ lower order terms. Show that the ideal I is generated by the elements $f_{j,k}$ for $j = 0, \ldots, N$ and $k = 1, \ldots, n_i$ as follows: use induction on the degree d to prove that any polynomial $f \in I$ of degree d is in the ideal generated by the $f_{j,k}$'s.

Problem 6.12. Let k be a field.

(a) Use the generalized form of Eisenstein's criterion to show that $xy - zw \in k[x, y, z, w]$ is irreducible.
(b) Show that $k[x, y, z, w]/(xy - zw)$ is an integral domain. (You may use the fact proven in lecture that $k[x, y, z, w]$ is a UFD.)
(c) Show that $k[x, y, z, w]/(xy - zw)$ is not a UFD.