Math 403B: Introduction to Modern Algebra, Winter Quarter 2018
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Homework 5
Due: Wednesday, February 14

Problem 5.1. Judson 17.4.20

Problem 5.2. Judson 17.4.21

Problem 5.3. Judson 17.4.24

Problem 5.4. Judson 17.4.26

Problem 5.5. Find a formula for the number of polynomials of degree less than or equal to N in $\mathbb{Z}/2[x]$. Use this to prove there are irreducible polynomials of arbitrarily large degree in this polynomial ring.

Problem 5.6. Sometimes we can prove irreducibility of a polynomial in $\mathbb{Q}[x]$ using a technique called "reduction mod p." Suppose that $f(x) \in \mathbb{Z}[x]$ is a polynomial of positive degree and $p \in \mathbb{Z}$ is a prime integer not dividing the highest degree coefficient of $f(x)$. Reduce the polynomial modulo p to get a polynomial $f_p(x) \in \mathbb{Z}/p[x]$. Prove that if $f_p(x)$ is irreducible in $\mathbb{Z}/p[x]$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Problem 5.7. Use Problem 5.6 to determine whether the following polynomials are irreducible:
(a) $x^5 + x^2 + 2 \in \mathbb{Q}[x]$; and
(b) $x^5 + x^4 + 2x^2 + 2x + 2 \in \mathbb{Q}[x]$.

Problem 5.8. Determine whether or not $x^3 + 2x + 1$ and $x^4 + 3x - 2$ are congruent modulo the ideal $(x^2 + 2x + 2) \subset \mathbb{Q}[x]$.

Problem 5.9.
(a) Find a constant polynomial congruent to $x^3 - x + 1$ modulo the ideal $(x + 2) \subset \mathbb{Q}[x]$.
(b) Find a polynomial of degree < 3 congruent to $x^7 + x + 1$ modulo the ideal $(x^3 + x + 1) \subset \mathbb{Z}/3[x]$.
(c) Find a polynomial of degree < 2 congruent to $x^4 + 2x + 4$ modulo the ideal $(x^2 + 1) \subset \mathbb{Z}/5[x]$.

Problem 5.10. List a complete set of representatives of $\mathbb{Z}/2[x]/(x^4 + x^2 + 1)$.

Problem 5.11. Let $p = x^2 + 1 \in \mathbb{Z}/3[x]$. Write down the multiplication table for $\mathbb{Z}/3[x]/(p)$.

Problem 5.12. Find a field with 32 elements and another with 27 elements.