MATH 403 Winter 2018

Homework III

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Problem 3.1 Let $\pi : R \to R/I$ be the quotient map. Let $J \subset R$ be an ideal containing I. Then J is an abelian group. Since π is a homomorphism of abelian groups, $\pi(J) \subset R/I$ is an abelian group. Let us check $\pi(J)$ is an ideal. If $r \in R$ and $j \in J$, then $(r+I)\cdot\pi(j) = r \cdot j + I$. Since J is an ideal, we see that $r \cdot j \in J$ so that $(r+I)\cdot\pi(j) \in \pi(J)$. This shows that $\pi(J)$ is an ideal.

Conversely, if $\overline{J} \subset R/I$ is an ideal, we consider $\pi^{-1}(\overline{J}) \subset R$. You can easily show that this is an ideal of R. Since $\pi(I) = 0 \in \overline{J}$ we see that $\pi^{-1}(\overline{J})$ necessarily contains I.

Now check that $\pi^{-1}(\pi(J)) = J$ for any ideal $J \subset R$ containing I and $\pi(\pi^{-1}(\overline{J})) = \overline{J}$ for any ideal $\overline{J} \subset R/I$. For the first, if $r \in R$ with $\pi(r) = \pi(j)$ for some $j \in J$. Then $r - j \in I \subset J$ so that $r \in J$. We get

$$\pi^{-1}(\pi(J)) \subset J.$$

But $J \subset \pi^{-1}(\pi(J))$ by definition of inverse image. The second equality follows from surjectivity of π.

Problem 3.3 Let us show that the units in $\mathbf{Z}_{(p)}$ is of the form $\frac{a}{b}$ where $p \nmid a$ and $(a,b) = 1$. If $\frac{a}{b}$ is a unit, then there exists $\frac{c}{d} \in \mathbf{Z}_{(p)}$ such that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = 1.$$

This means $ac = bd$ in $\mathbf{Z}$. If $p \mid a$, then $p \mid bd$. Since $p \nmid d$ by construction, we have $p \mid b$. This contradicts $(a,b) = 1$. Hence if $\frac{a}{b}$ is a unit with $(a,b) = 1$, $p \nmid a$. Conversely, if $(a,b) = 1$ and $p \nmid a$, then $\frac{b}{a} \in \mathbf{Z}_{(p)}$ is a unit for $\frac{a}{b}$.

Next, oberve that an ideal in a commutative ring with identity is proper if and only if it does not contain any unit. In our case the set of non-units $p\mathbf{Z}_{(p)} := \{\frac{a}{b} | (a,b) = 1 \text{ and } p \mid a\}$ obviously forms an ideal in $\mathbf{Z}_{(p)}$. Hence any proper ideal is inside $p\mathbf{Z}_{(p)}$. We conclude that there is only one maximal ideal.

Problem 3.5

1. By assumption ther exists $i_1, i_2 \in I$ and $j_1, j_2 \in J$ such that $i_1 + j_1 = r$ and $i_2 + j_2 = s$. Let $x = i_2 + j_1$, we see that $x - r \in I$ and $x - s \in J$.

2. If $x_1 - r \in I$, $x_2 - r \in I$ and $x_1 - s \in J$, $x_2 - s \in J$, we have that

$$x_1 - x_2 = (x_1 - r) - (x_2 - r) = (x_1 - s) - (x_2 - s) \in I \cap J.$$

3. The projections $\pi_I : R \to R/I$ and $\pi_J R \to R/J$ induces a ring hmomorphism $\pi : R \to R/I \times R/J$. Concretely, $\pi(r) = (r + I, r + J)$. By part (a), π is surjective. The kernel is obviously $I \cap J$. Hence we have an isomoprhism

$$R/(I \cap J) \simeq R/I \times R/J.$$

Problem 3.6

1. Let $r_1, r_2 \in R$ and $r_1 \cdot r_2 \in \phi^{-1}(\mathfrak{p})$. Then $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, we see that either $\varphi(r_1)$ or $\varphi(r_2)$ is in $\mathfrak{p}$. This means either r_2 or r_1 is in the preimage.

2. Consider the inclsion $\mathbf{Z} \hookrightarrow \mathbf{Q}$. 0 is a maximal ideal in $\mathbf{Q}$ but its preimage is not a maximal ideal in $\mathbf{Z}$.

Problem 3.9

1. No roots.
2. 2.
3. 3,4.
4. No roots.