Midterm solutions

Advanced Linear Algebra (Math 340)
Instructor: Jarod Alper
April 26, 2017 **Name:** ____________________

Read all of the following information before starting the exam:

- You may not consult any outside sources (calculator, phone, computer, textbook, notes, other students, ...) to assist in answering the exam problems. All of the work will be your own!
- Show all work, clearly and in order, if you want to get full credit. Partial credit will be awarded.
- Throughout the exam, the symbol $\mathbb{F}$ denotes either the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.
- Circle or otherwise indicate your final answers.
- Good luck!

Problem	Points
1 (10 points)	______
2 (10 points)	______
3 (10 points)	______
4 (10 points)	______
5 (10 points)	______
Total (50 points)	

1. (*10 points*)

Determine whether the following statements are true or false. It is not necessary to explain your answers.

(1) False______ If V is a vector space over $\mathbb{F}$, then any equality of the form $ax = bx$ where $a, b \in \mathbb{F}$ and $x \in V$ implies that in fact $a = b$.

(2) True______ If S is a linearly independent subset of a vector space V, then any subset of S is also linearly independent.

(3) True______ Every subspace of $\mathbb{R}^n$ is finite dimensional.

(4) True______ If $T \colon V \to W$ is a linear transformation, then $T(0) = 0$.

(5) False______ If $T \colon V \to W$ is a linear transformation, then $N(T) = (0)$ if and only if T is onto.

2. (*10 points*) Recall that $P_n(\mathbb{R})$ denotes the vector space consisting of polynomials of degree $\leq n$ with real coefficients.

a. (*5 pts*) Show that that $\beta = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is a basis of $P_3(\mathbb{R})$

Solution: As $\{1, x, x^2, x^3\}$ is a basis of $P_3(\mathbb{R})$, we know that $\dim P_3(\mathbb{R}) = 4$. Therefore, it suffices to show that the span of β is all of $P_3(\mathbb{R})$. Indeed, we know from lecture that there is a linearly independent subset of β which has the same span and therefore is a basis. Since the number of elements in any two bases is the same, we see that as long as the span of β is $P_3(\mathbb{R})$, then β must be a basis.

To show that β spans $P_3(\mathbb{R})$, it suffices to show that $1, x, x^2$ and x^3 can be written as linear combinations of elements of β. Clearly, 1 is in the span of β. Also, $x = -1 + (1+x) \in \text{span}(\beta)$, $x^2 = (1+x+x^2) + -1(1+x) \in \text{span}(\beta)$, and $x^3 = (1+x+x^2+x^3) + -1(1+x+x^2) \in \text{span}(\beta)$.

b. (*5 pts*) What is the nullspace of the linear transformation $T\colon P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by taking a polynomial $f(x)$ to its derivative $\frac{df}{dx}$?

Solution: Let $f = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{R})$. Then $T(f) = a_1 + 2a_2x + 3a_3x^2$. We see that $T(f) = 0$ if and only if $a_1 = a_2 = a_3 = 0$. We conclude that

$$N(T) = \{a_0 \mid a_0 \in \mathbb{R}\}.$$

In other words, the null space consists of only the constant polynomials.

3. (*10 points*) Suppose that V is a vector space over $\mathbb{F}$ and that $\{u, v\}$ is a basis of V.

a. (*3 pts*) What is the dimension of V?

Solution: The dimension of V is the number of elements in any basis. Therefore, $\dim V = 2$.

b. (*4 pts*) Show that $\{u - v, u + v\}$ is also a basis of V.

Solution: To see that $\{u-v, u+v\}$ is linearly independent, suppose $a(u-v)+b(u+v) = 0$ for some $a, b \in \mathbb{F}$. Then $(b+a)u+(b-a)v = 0$ and since $\{u, v\}$ is a basis, we see that $b+a = b-a = 0$. This clearly implies that $a = b = 0$.

To see that $\{u - v, u + v\}$ spans V, it suffices to show that $u, v \in \operatorname{span}(u - v, u + v)$. But clearly $u = \frac{1}{2}(u - v) + \frac{1}{2}(u + v)$ and $v = -\frac{1}{2}(u - v) + \frac{1}{2}(u + v)$.

c. (*3 pts*) If $\{u, v, w\}$ is a basis of a vector space W over $\mathbb{F}$, then is $\{u - v, v - w, w - u\}$ also a basis of W?

Solution: No, $\{u - v, v - w, w - u\}$ is not linearly dependent as $(u - v) + (v - w) + (w - u) = 0$.

4. (*10 points*)

a. (*5 pts*) Let V be a vector space over $\mathbb{F}$. Show that if W_1, W_2 are subspaces of V, then so is the intersection $W_1 \cap W_2$.

By definition, the intersection $W_1 \cap W_2$ consists of those vectors in V that lie both in W_1 and W_2.

Solution: We need to show the following two properties

- For $w_1, w_2 \in W_1 \cap W_2$, then $w_1 + w_2 \in W_1 \cap W_2$: Since w_1, w_2 are in both W_1 and in W_2 and using that both W_1 and W_2 are subspaces, we see that $w_1 + w_2$ are contained in W_1 and W_2. Therefore, $w_1 + w_2 \in W_1 \cap W_2$.
- For $w \in W_1 \cap W_2$ and $a \in \mathbb{F}$, then $aw \in W_1 \cap W_2$: Since w is in both W_1 and W_2 and since both W_1 and W_2 are closed under scalar multiplication, we see that aw is contained in both W_1 and W_2; that is, $aw \in W_1 \cap W_2$.

b. (*5 pts*) Let $U, W \subset V$ be supsaces of a vector space V. Denote by $T \colon V \to V/W$ the linear transformation to quotient space V/W. Show that U is contained in W if and only if $T(U) = (0)$.

Recall that V/W denotes the vector space of cosets $v + W$ for $v \in V$, and that T is defined by setting $T(v) = v + W$. Also, the set $T(U) \subset V/W$, by definition, consists of all vectors of the form $T(u)$ for some $u \in U$.

Solution: The zero element in V/W is the coset $W \in V/W$. Since $T(U) = \{u + W \mid u \in U\}$, we see that $T(U) = (0)$ if and only if $u + W = W$ for all $u \in U$. But we know that a coset $v + W$ is equal to W if and only if $v \in W$. We thus see that $T(U) = (0)$ if and only if for all $u \in U$, then $u \in W$. This latter condition is simply the requirement that $U \subset W$.

5. (*10 points*) Let V be a vector space over $\mathbb{F}$. Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a subset of V. Define the linear transformation $T\colon \mathbb{F}^n \to V$ by setting $T(a_1, a_2, \dots, a_n) = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. Prove that β is a basis if and only if T is an isomorphism.

Solution: We will prove the following two equivalences:

- T is one-to-one if and only if β is linearly independent: We know that T is one-to-one if and only if $N(T) = (0)$. Suppose $w = (a_1, a_2, \dots, a_n) \in \mathbb{F}^n$ is a vector such that $T(w) = 0$. Using the definition of the linear transformation T, we see that $T(w) = a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. Thus, $N(T) = (0)$ if and only if for all scalars $a_1, a_2, \dots, a_n$ with $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. This last property is the definition that β is linearly independent.
- T is onto if and only if β spans V: By definition, T is onto if and only if the range $R(T) = V$. But as $R(T) = \{T(w) \mid w \in \mathbb{F}^n\}$, we see that T is onto if and only if for every vector $v \in V$, there exists $w = (a_1, a_2, \dots, a_n) \in \mathbb{F}^n$ such that $T(w) = a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$. This last property is the definition that β spans V.

Finally, we know that β is a basis if and only if β is both linearly independent and spans V. Similarly, T is an isomorphism if and only if T is one-to-one and onto. By the two equivalences above, we conclude that β is a basis if and only if T is an isomorphism.