Math 300: Wed 5/18

Today: Finishing Ch. 4

Review

Defn A set S is called

① countable if S is finite or $S \cong \mathbb{N}$

② uncountable if S is not countable.

Defn For sets A and B, we say

① $\#A \leq \#B$ if $\exists$ injection $f: A \to B$

② $\#A = \#B$ if $\exists$ bijection $f: A \to B$

③ $\#A < \#B$ if $\#A \leq \#B$ and $\#A \neq \#B$

Thm (Schröder-Bernstein)

$\#A \leq \#B$ & $\#B \leq \#A \Rightarrow \#A = \#B$

Cantor's Thm 1 For any set A

$$\#A < \#P(A)$$

Example:

If A is finite of size n,

$$\#P(A) = 2^n > n = \#A$$

Example:

$$\#\mathbb{N} < \#P(\mathbb{N})$$

$\Rightarrow$ $P(\mathbb{N})$ uncountable

Cantor's Thm 2

$\mathbb{R}$ is uncountable.

Recall $\underbrace{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}}_{\text{all countable}} \subseteq \mathbb{R}$

Pf: Know $\mathbb{R} \simeq (0,1)$

Reason: Use $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ bijective

$\rightsquigarrow$ Suffices to show $(0,1)$ uncountable

Pf by contradiction: assume $\exists$ bijection $f : \mathbb{N} \longrightarrow (0,1)$

$$1 \longmapsto 0.a_{11}a_{12}a_{13}\cdots$$

decimal digits $a_{in} \in \{0,\dots,9\}$

$$2 \longmapsto 0.a_{21}a_{22}a_{23}\cdots$$

$$3 \longmapsto 0.a_{31}a_{32}a_{33}\cdots$$

$\vdots$

Have $a_{ij} \in \{0,1,2,\dots,9\}$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$

Consider

$$m = 0.m_1m_2m_3m_4\cdots\cdots$$

where $m_i = \begin{cases} 2 & a_{ii} = 1 \\ 1 & a_{ii} \neq 1 \end{cases}$

Since $m \in (0,1)$, $\exists\, i \in \mathbb{N}$ s.t.

$$\boxed{f(i) = m}$$

Means

$$0.a_{i1}a_{i2}\cdots a_{ii}\cdots\cdots = 0.m_1m_2m_3\cdots$$

$\Longrightarrow a_{ij} = m_j \;\; \forall j$

$\Longrightarrow \boxed{a_{ii} = m_i}$ but $m_i \neq a_{ii}$

Contradiction.

Note

$$\mathbb{R} = \mathbb{Q} \cup \{\text{irrational number}\}$$

$\mathbb{R}$: uncountable

$\mathbb{Q}$: countable

$\Longrightarrow$ {irrational numbers} uncountable

Hint/background for HW8

↳ Fundamental Thm of Arithmetic
(Thm 6.29 in Gersten)

For any integer $n > 1$,
$\exists$ primes $p_1, \ldots, p_r$ such that

$n = p_1 p_2 \cdots p_r$.

Moreover, this is unique in the following sense:

if $n = q_1 \cdots q_s$ is another factorization into primes, then

(1) $r = s$

(2) after reordering p_i's are same as q_i's

(More precisely, $\forall i \ \exists! j \ \ p_i = q_j$)

Ex: $38 = 2 \cdot 19 = 19 \cdot 2$

Hint: Find all $n \in \mathbb{N}$ such that

n^2+1 is divisible by $n+1$.

(that is, $(n+1) \mid (n^2+1)$.)

If $n+1$ divides some integer A, then $n+1$ divides $(n^2+1) \pm A$

Choose A cleverly.

$A = (n+1)(n-1)$