MATH 300 - Introduction to Mathematical Reasoning, Spring 2022 - Homework 3 Solution

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Section 2.4 Q4

a) $A \subset B \Longrightarrow B \not\subset A$.

Solution: I feel there are two ways to go about this. Either you do a proof by elements or you try working directly with sets.

Option 1.

Proof. Suppose that $A \subset B$. Then we have $\forall x \in A : x \in B$ and $\exists x \in B : x \notin A$. But the proposition with $\exists$ is precisely the definition of $B \not\subseteq A$. Also $B \not\subseteq A \Rightarrow B \not\subset A$. (To convince yourself of this think about the contrapositive statement.) □

Option 2.

Proof. Suppose that $A \subset B$. Then by definition $A \subseteq B$ and $A \neq B$. Now suppose by contradiction that $B \subset A$. Then in particular, $B \subseteq A$ and by Theorem 2.17 we would have $A = B$. This contradicts the fact that the containment is strict ($A \neq B$). □

- $A \subseteq \emptyset \Longrightarrow A = \emptyset$.

Solution: Again you can think about it in terms of elements or sets.

Option 1a.

Proof. Suppose that $A \subseteq \emptyset$. Then $\forall x \in A : x \in \emptyset$. But $x \in \emptyset$ is false for all x. The only way that the "for all" statament can be true is that it is **vacuously true**. That is if the set A is empty. □

Option 1b.

Proof. We can prove the contrapositive. Suppose that $A \neq \emptyset$. Then there exists $x \in A$. Since $x \notin \emptyset$ we conclude that $A \not\subseteq \emptyset$. □

Option 2.

Proof. Suppose that $A \subseteq \emptyset$. Also for any set A it is true that $\emptyset \subseteq A$ (Theorem 2.14). Therefore, $A = \emptyset$.

□

While grading I encountered yet another clever way of proving this assuming we know what the power set of the empty set is.

Option 3.

Proof. Suppose that $A \subseteq \emptyset$. Then $A \in P(\emptyset)$. But $P(\emptyset) = \{\emptyset\}$ is a singleton (only has one element). Then $A = \emptyset$. $\square$

However, you could say that it should be the other way around. The fact that $P(\emptyset) = \{\emptyset\}$ follows by the statement that we are proving here.

Section 2.4 Q8

Solution: A good way to prove an equivalence is to work on each implication (or direction if you are thinking about arrows) individually.

Proof. First let us show the only if statement (or $\Rightarrow$ direction). Suppose that $A \subseteq B$. Let S be an arbitrary subset of A. Then we have $T \subseteq A \subseteq B$. Since the relation $\subseteq$ is transitive we conclude that $T \subseteq B$.

The if statement ($\Leftarrow$ direction) is actually very simple. Suppose that all subsets of A are subsets of B. Then in particular, A is a subset of A (Theorem 2.14). Therefore, $A \subseteq B$. $\square$

If you don't believe that the $\subseteq$ relation is transitive you can always prove it using elements: $\forall x \in T : x \in A \wedge \forall x \in A : x \in B \Longrightarrow \forall x \in T : x \in B$.

Section 2.4 Q10

Solution: The statement $\{\{\{5\}\}\} = \{\{5\}\}$ is false. How can we show it? By checking if the elements of these sets are the same.

Proof. Let $A = \{\{\{5\}\}\}$ and $B = \{\{5\}\}$. Then $\{\{5\}\} \in A$ but $\{\{5\}\} \notin B$. $\square$

Many people finish the proof here and I did not take points of for it. However, notice that this is assuming implicitly the following statement $\{\{5\}\} \neq \{5\}$. In my opinion the question of whether this is true or not is as valid as our original one and deserves a proof of its own. Some people did notice this and came up with this very neat proof.

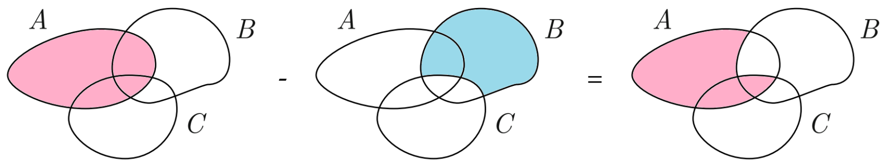
Proof. Consider the following chain of equivalences

$$\{\{\{5\}\}\} = \{\{5\}\} \Longleftrightarrow \{\{5\}\} = \{5\} \Longleftrightarrow \{5\} = 5.$$

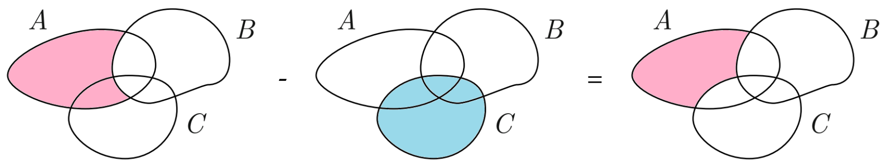
The last statement is clearly false because a number is not a set. It follows that the rest of the statements are false as well. $\square$

Section 2.5 Q6

Solution: $A - (B - C)$:



$(A - B) - C$:



As we can see from the Venn diagrams these two sets are different. Many people pointed out that the difference is precisely $A \cap C$. So any example where $A \cap C \neq \emptyset$ can work as a counter example.

This is the simplest possible one (in terms of set sizes at least).

Take $A = C = \{1\}$, $B = \emptyset$.

Then

$$\begin{aligned} A - (B - C) &= \{1\} - (\emptyset - \{1\}) \\ &= \{1\} - \emptyset \\ &= \{1\} \end{aligned}$$

But

$$\begin{aligned} A - (B - C) &= (\{1\} - \emptyset) - \{1\}) \\ &= \{1\} - \{1\} \\ &= \emptyset \end{aligned}$$

Section 2.5 Q8

Solution:

a) $A \oplus A = \emptyset$.

Proof. $A \oplus A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$. □

b) $A \oplus \emptyset = A$.

Proof. $A \oplus \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$. □

c) $(A \oplus B) - (A - B) = B - A$.

Proof.

$$\begin{aligned} x \in (A \oplus B) - (A - B) &\iff (x \in A - B \vee x \in B - A) \wedge x \notin A - B. \\ \text{(distributive law)} &\iff (x \in A - B \wedge x \notin A - B) \vee (x \in B - A \wedge x \notin A - B) \\ &\iff \mathrm{F} \vee (x \in B - A \wedge x \notin A - B) \\ &\iff x \in B - A \wedge x \notin A - B \\ &\implies x \in B - A. \end{aligned}$$

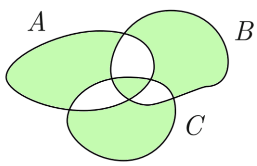
On the other hand $x \in B - A \iff x \in B \wedge x \notin A \implies x \notin A - B$. Because of this we can replace the last implication by an equivalence. □

d) $A \oplus B = \emptyset \implies A = B$.

Proof. (By contrapositive) Suppose that $A \neq B$. Then either there exists x such that $x \in A$ but $x \notin B$ or there exists x such that $x \in B$ but $x \notin A$. In the first case $x \in A - B$ while in the second $x \in B - A$. In either case $x \in A \oplus B$ so we conclude that $A \oplus B$ is nonempty. □

e) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

The Venn diagram in both cases looks something like this.



As most of you might have notice, a formal proof of this is very involved. If you want to see a very thorough exposition of it I invite you to check the following notes: http://ramanujan.math.trinity.edu/rdaileda/teach/s20/m3326/symmetric.pdf.

Section 2.6 Q4

Solution:

a) Observe that in this case the index set is $I = \{1, 2, 3\}$. In particular,

$$\bigcap_{i \in I} A_i = A_1 \cap A_2 \cap A_3.$$

Now, if $A_1 = \{1, 2, 3, 4\}, A_2 = \{3, 4, 5, 6\}$ and $A_3 = \{1, 6, 7\}$ then $A_1 \cap A_2 = \{3, 4\}$ and we conclude that the family $\{A_1, A_2, A_3\}$ is not pairwise disjoint.

However, $A_1 \cap A_2 \cap A_3 = \{3, 4\} \cap \{1, 6, 7\} = \emptyset$. Therefore the family is disjoint.

b) *Proof.* Suppose that the family $\{A_1, A_2, A_3\}$ is pairwise disjoint. Then in particular, $A_1 \cap A_2 = \emptyset$. Therefore,

$$A_1 \cap A_2 \cap A_3 = \emptyset \cap A_3 = \emptyset.$$

This concludes that the family is disjoint. $\square$

Section 2.6 Q6

Solution:

a) $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{Z}$.

Some key observations. 1) All M_n are subsets of $\mathbb{Z}$. 2) M_1 is a member of our family of sets. 3) $M_1 = \mathbb{Z}$. These three facts combined imply that $\bigcup_{n\in\mathbb{N}} M_n = M_1 = \mathbb{Z}$.

b) $\bigcap_{n\in\mathbb{N}} M_n = \{0\}$.

The key observations for this result are the following. 1) For all $n \in \mathbb{N}$, $0 \in M_n$. 2) M_0 is a member of our family of sets. 3) $M_0 = \{0\}$.

c) $\bigcup_{p \text{ prime}} M_n = \mathbb{Z} - \{-1, 1\}$.

This follows from the unique factorization theorem in arithmetic. This states that any number greater than 1 can be represented uniquely as a product of prime factors. In particular, if $m > 1$ and p is a prime factor of m then $m \in M_p$. For negative numbers less than -1 the same argument applies, since m and $-m$ are divided by the same prime factors. Finally, it is easy to check that no prime number divides 1 or -1. The only divisors of these numbers are again 1 and -1 which are not prime. Last but not least 0 is contained in all M_p so it is contained in the union as well.

d) $\bigcap_{p \text{ prime}} M_n = \{0\}$.

Suppose that $|m| > 0$, then we can find a prime number p such that $p > |m|$. It follows that p does not divide m so $m \notin M_p$. The fact we are using implicitly is that the set of prime numbers is infinite.

e) $\bigcup_{n\geq 6} M_n = \mathbb{Z} - \{-5, -4, -3, -2, -1, 1, 2, 3, 4, 5\}$.

Observe that $\{-n, n\} \subseteq M_n$. The numbers that are missing are those such that $0 < |m| < 6$. There is no $n \geq 6$ such that n divides m.

f) $\bigcup_{n\in M_5} M_n = M_5$.

Some key facts to note here. 1) If $n \in M_5$ then $M_n \subseteq M_5$. 2) $5 \in M_5$ which implies that M_5 is one of the sets in the family that we are taking the union of.

Final note: These types of sets are known as ideals in abstract algebra. You might see them again in your math studies.

Section 2.7 Q2

Solution:

a) $P(\{2\}) = \{\emptyset, \{2\}\}$.

b) $P(P(\{2\})) = \{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\emptyset, \{2\}\}\}$.

c) $P(P(P(\{2\}))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{2\}\}\}, \{\{\emptyset, \{2\}\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\{2\}\}\}, \{\emptyset, \{\emptyset, \{2\}\}\}, \{\{\emptyset\}, \{\{2\}\}\}, \{\{\emptyset\}, \{\emptyset, \{2\}\}\}, \{\{\{2\}\}, \{\emptyset, \{2\}\}\}, \{\emptyset, \{\emptyset\}, \{\{2\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{2\}\}\}, \{\emptyset, \{\{2\}\}, \{\emptyset, \{2\}\}\}, \{\{\emptyset\}, \{\{2\}\}, \{\emptyset, \{2\}\}\}, \{\emptyset, \{\emptyset\}, \{\{2\}\}, \{\emptyset, \{2\}\}\}.\}$

Section 2.7 Q4

Solution:

Proof. We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By Theorem 2.36.b, $P(A) \subseteq P(A \cup B)$ and $P(B) \subseteq P(A \cup B)$. It follows that $P(A) \cup P(B) \subseteq P(A \cup B)$.

To show that the containment is strict observe that by our hypothesis there exists $x \in A - B$ and $y \in B - A$. Then we have $\{x, y\} \in P(A \cup B)$ but $\{x, y\} \notin P(A) \cup P(B)$. □

Section 2.7 Q8

Solution: One way of describing $P(A \cup \{x\})$ is the following:

$$P(A \cup \{x\}) = P(A) \cup \{S \cup \{x\} : S \in P(A)\}.$$

What this is saying is that $P(A \cup \{x\})$ consist of two copies of $P(A)$. One is left as it is while to the other we add the element x to each of the sets in the copy. For finite sets we can say that we are duplicating $P(A)$.