Midterm 1 Solutions

Calculus I (Math 124)
Instructor: Jarod Alper
Fall 2019
October 22, 2019 **Name:** ________________

Section: ________________

Read all of the following information before starting the exam:

- You may use a Ti-30x IIS calculator.
- You may refer to your hand-written note sheet (8.5"x11", two-sided).
- You may not consult any other outside sources (phone, computer, textbook, other students, ...) to assist in answering the exam problems. All of the work will be your own!
- Write clearly!! You need to write your solutions carefully and clearly in order to convince me that your solution is correct. Partial credit will be awarded.
- Good luck!

Problem	Points
1 (20 points)	______
2 (20 points)	______
3 (20 points)	______
4 (20 points)	______
5 (20 points)	______
Total (100 points)	

Problem 1. Calculate the following limits:

(a)

$$\lim_{x\to 1} \frac{x^3 - x^2}{x^2 - 1}$$

Solution: By factoring $x^3 - x^2 = x^2(x-1)$ and $x^2 - 1 = (x-1)(x+1)$, we compute

$$\begin{aligned}\lim_{x\to 1} \frac{x^3 - x^2}{x^2 - 1} &= \lim_{x\to 1} \frac{x^2(x-1)}{(x-1)(x+1)} \\ &= \lim_{x\to 1} \frac{x^2}{x+1} \\ &= \frac{1}{2}\end{aligned}$$

(b)

$$\lim_{x\to 2^+} \frac{\cos(\pi x)}{8 - x^3}$$

Solution: First observe that $\lim_{x\to 2^+} \cos(\pi x) = 1$ and that $\lim_{x\to 2^+} = 0$. Thus the limit $\lim_{x\to 2^+} \frac{\cos(\pi x)}{8-x^3}$ will either be positive or negative infinity. To compute the sign, observe that as x approaches 2 from the right, the values of $\cos(2\pi)$ are positive but the values of $8-x^3$ are negative. It follows that $\lim_{x\to 2^+} \frac{\cos(\pi x)}{8-x^3} = -\infty$.

(c)

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+1}}$$

Solution: We will divide both the numerator and denominator by $1/x$ to compute

$$\begin{aligned}\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \left(\frac{x+1}{\sqrt{x^2+1}} \cdot \frac{1/x}{1/x} \right) \\ &= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}} \\ &= \frac{(\lim_{x\to\infty} 1) + (\lim_{x\to\infty} \frac{1}{x})}{\sqrt{(\lim_{x\to\infty} 1) + (\lim_{x\to\infty} \frac{1}{x^2})}} \\ &= \frac{1+0}{\sqrt{1+0}} \\ &= 1.\end{aligned}$$

Problem 2. A ball is thrown into the air at time $t = 0$ seconds and the height $h(t)$ of the ball from the ground after t seconds is given in meters by the equation

$$h(t) = -5t^2 + 20t + 10.$$

(a) Find the height of the ball when it is released.

Solution: We calculate $h(0) = 10$ meters.

(b) Find the average velocity of the ball between the time it is released and the time it hits the ground.

Solution: To find the time that the ball hits the ground, we solve for the solutions to $h(t) = -5t^2 + 20t + 10 = 0$. The solutions are given by the quadratic formula

$$t = \frac{-20 \pm \sqrt{20^2 - 4(-5)(10)}}{2(-5)} = \frac{-20 \pm \sqrt{600}}{-10} = 2 \pm \sqrt{6} = -0.449, 4.449$$

Thus the ball hits the ground at $t = 2 + \sqrt{6} = 4.449$ seconds.

The height of the ball at $t = 0$ is $h(0) = 10$ and at $t = 2 + \sqrt{6} = 4.449$ is 0. Thus the average velocity, which is the change in height over the change in time, is given by

$$-10/(2 + \sqrt{6})\, m/s = -10/4.449\, m/s = -2.247\, m/s.$$

(c) Find the velocity of the ball when it hits the ground.

Solution: The velocity at time t is given by the derivative $h'(t) = -10t + 20$. Evaluating at $t = 2 + \sqrt{6} = 4.449$, we get that the velocity is $-10\sqrt{6}$ m/s or -24.495 m/s.

Problem 3.

(a) Find the equation of the tangent line to the graph of

$$y = \frac{x^2+1}{x-3}$$

at $(4, 17)$.

Solution: We first compute the derivative using the quotient rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dx}(x^2+1)(x-3) - (x^2+1)\frac{d}{dx}(x-3)}{(x-3)^2} \\ &= \frac{2x(x-3)-(x^2+1)}{(x-3)^2} \\ &= \frac{x^2-6x-1}{(x-3)^2} \end{aligned}$$

Evaluating at $x = 4$, we get that $\frac{dy}{dx} = -9$. Therefore, the equation of the tangent line through $(4, 17)$ is

$$y = -9(x-4) + 17 = -9x + 53.$$

(b) Find the x-coordinates of all of the points on the graph where the tangent line is horizontal.

Solution: We simply need to find the x-coordinates of all the points where $\frac{dy}{dx} = 0$. Solutions of

$$\frac{dy}{dx} = \frac{x^2-6x-1}{(x-3)^2} = 0$$

are given by solutions of the numerator $x^2 - 6x - 1 = 0$. Using the quadratic formula, we see that there are two solutions $(6 \pm \sqrt{40})/2 = 3 \pm \sqrt{10}$, or in other words -0.1623 and 6.1623.

Problem 4. Let $f(x) = \sqrt{x^2+5}$. Find $f'(2)$ using the definition of the derivative as a limit.

Solution: Using the definition, we compute

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(2+h)^2+5}-3}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h^2+4h+9}-3}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h^2+4h+9}-3}{h} \cdot \left(\frac{\sqrt{h^2+4h+9}+3}{\sqrt{h^2+4h+9}+3}\right) \\
&= \lim_{h\to 0} \frac{(h^2+4h+9)-9}{h(\sqrt{h^2+4h+9}+3)} \\
&= \lim_{h\to 0} \frac{h^2+4h}{h(\sqrt{h^2+4h+9}+3)} \\
&= \lim_{h\to 0} \frac{h+4}{(\sqrt{h^2+4h+9}+3)} \\
&= \frac{0+4}{\sqrt{9}+3} \\
&= 2/3.
\end{aligned}$$

We can check the answer using our rules for derivatives. Namely, writing $f(x) = (x^2+5)^{1/2}$ and using the chain rule, we compute that $f'(x) = \frac{1}{2}(x^2+5)^{-1/2}(2x)$. Evaluating at $x=2$, we see that $f'(2) = 2/3$ confirming our answer above.

Problem 5. Find the derivatives of the following functions:
(a) $f(x) = \frac{2\tan(x)}{x^3+1}$.

Solution: Using that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$, we compute using the quotient rule that.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{2\tan(x)}{x^3+1}\right) \\
&= \frac{\frac{d}{dx}(2\tan(x))(x^3+1) - 2\tan(x)\frac{d}{dx}(x^3+1)}{(x^3+1)^2} \\
&= \frac{2\sec^2(x)(x^3+1) - 2\tan(x)(3x^2)}{(x^3+1)^2} \\
&= \frac{2\sec^2(x)(x^3+1) - 6x^2\tan(x)}{(x^3+1)^2}
\end{aligned}$$

(b) $f(x) = e^{\sqrt[3]{x}\sin(x)}$.

Solution: We compute that:

$$\begin{aligned}
f'(x) &= \tfrac{d}{dx}\left(e^{\sqrt[3]{x}\sin(x)}\right) \\
&= e^{\sqrt[3]{x}\sin(x)}\tfrac{d}{dx}(\sqrt[3]{x}\sin(x)) && \text{(chain rule)} \\
&= e^{\sqrt[3]{x}\sin(x)}\left(\tfrac{d}{dx}(\sqrt[3]{x})\sin(x) + \sqrt[3]{x}\tfrac{d}{dx}(\sin(x))\right) && \text{(product rule)} \\
&= e^{\sqrt[3]{x}\sin(x)}\left(\tfrac{1}{3}x^{-2/3}\sin(x) + \sqrt[3]{x}\cos(x)\right)
\end{aligned}$$