Algebra 2, Semester 1 2015
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Tutorial 6
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Tutorial 6.1. Determine the Galois group over $\mathbb{Q}$ of $x^8 - 3$.

Tutorial 6.2. Let L be the splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$. Give the complete correspondence between intermediate field extensions $\mathbb{Q} \subseteq L' \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

Tutorial 6.3. Let $K \subseteq L$ be a Galois field extension with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Show that there exists elements $\alpha, \beta \in L$ such that $\alpha^2, \beta^2 \in K$ and $L \cong K(\alpha, \beta)$.

Tutorial 6.4. Find a *non-Galois* field extension $K \subseteq L$ such that the fundamental theorem of Galois theory fails; that is, there is not a bijective correspondence between intermediate field extensions $K \subseteq L' \subseteq L$ and subgroups $H \subseteq \mathrm{Gal}(L/K)$.

Tutorial 6.5. Consider the field $\mathbb{C}(t)$. As usual, we denote $\mathrm{Gal}(\mathbb{C}(t)/\mathbb{C})$ as the group of all $\mathbb{C}$-automorphisms $\sigma\colon \mathbb{C}(t) \to \mathbb{C}(t)$.

(a) Let $\sigma\colon \mathbb{C}(t) \to \mathbb{C}(t)$ be the field automorphism defined by $\sigma(t) = 1 - t$. Compute the fixed field $\mathbb{C}(t)^{\langle \sigma \rangle}$ of the subgroup $\langle \sigma \rangle \subseteq \mathrm{Gal}(\mathbb{C}(t)/\mathbb{C})$ generated by σ.
(b) Let $\tau\colon \mathbb{C}(t) \to \mathbb{C}(t)$ be the field automorphism defined by $\tau(t) = 1/t$. Compute the fixed field $\mathbb{C}(t)^{\langle \tau \rangle}$.
(c) Prove that $\sigma^2 = \tau^2 = \mathrm{id}$ and $(\sigma\tau)^3 = \mathrm{id}$. Conclude that the subgroup $G \subseteq \mathrm{Gal}(L/\mathbb{Q})$ generated by σ and τ is isomorphic to S_3.
(d) Show that

$$\mathbb{C}(t)^{\langle \sigma\tau \rangle} = \mathbb{C}(y), \qquad \text{where } y = \frac{t^3 - 3t + 1}{t(t-1)}.$$

(e) Show that $y + \sigma(y) = 3$. Conclude that

$$\mathbb{C}(t)^G = \mathbb{C}(y\sigma(y))$$