Algebra 2, Semester 1 2015
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Tutorial 3

Tutorial 3.1. Show $\mathbb{Q}[x]/(x^2-2) \cong \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Describe explicitly additional, multiplication and division on the right hand side.

Tutorial 3.2. Consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(a) What is the degree, $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$, of this field extension?
(b) Prove that this is a primitive field extension; that is, find an element α such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
(c) What is the minimum polynomial of the element α from part (b)? That is, find a monic polynomial with coefficients in $\mathbb{Q}$ of minimal degree which has α as a root.

Tutorial 3.3.

(a) Prove that $x^4 + 12x^3 + 9x^2 - 3x + 3$ is not irreducible.
(b) Prove that $x^m + 1 \in \mathbb{Q}[x]$ is irreducible if and only if $m = 2^n$.

Tutorial 3.4. Give an example of a polynomial $f(x) \in \mathbb{Z}[x]$ which has a root in every finite field $\mathbb{F}_p$, but no root in $\mathbb{Z}$. You may want to use the following remarkable fact from elementary number theory:

Given $a, b \in \mathbb{F}_p$, suppose that $x^2 = a$ and $x^2 = b$ do not have solutions in $\mathbb{F}_p$. Then $x^2 = ab$ does have a solution in $\mathbb{F}_p$.