Algebra 2, Semester 1 2015
Jarod Alper
Homework 7
Due: Monday, April 20

Problem 7.1.

(1) Show that $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{5}, i)$ is a Galois extension and compute its Galois group.
(2) Let ω be a primitive 7th root of unity. Show that $\mathbb{Q} \subseteq \mathbb{Q}(\omega)$ is a Galois extension and compute its Galois group.

Problem 7.2. Let L be the splitting field of $f(x) = x^3 - 2 \in \mathbb{Q}[x]$.

(1) There are three intermediate field extensions
$$\mathbb{Q} \subseteq K \subseteq L$$
with $|K : \mathbb{Q}| = 3$ and one such intermediate field extension with $|K : \mathbb{Q}| = 2$. Find these extensions.
(2) Compute the Galois group G of L over $\mathbb{Q}$.
(3) Find all subgroups $\{1\} \subsetneq H \subsetneq G$.

Problem 7.3. Let K be a field that is not characteristic 3. Suppose that $f(x) = x^3 - 3x + 1 \in K[x]$ is irreducible. Let $L = K(\alpha)$ where $f(\alpha) = 0$. Prove that f splits over L, and deduce that $K \subseteq L$ is a Galois extension with Galois group $\mathbb{Z}/3\mathbb{Z}$.

Hint: Factor f over L as $(x - \alpha)g$, and solve for the roots of g using the quadratic formula. Use the fact that $12 - 3\alpha^2 = (-4 + \alpha + 2\alpha^2)^2$ is a perfect square in $k(\alpha)$.