Algebra 2, Semester 1 2015
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Homework 10
Due: Monday, May 18

Problem 10.1. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 3. Let $\mathbb{Q} \subseteq L$ be the splitting field of $f(x)$ over $\mathbb{Q}$.

(a) Show that if $f(x)$ has only one real root, then $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.

(b) Recall that the discriminant Δ is defined as

$$\Delta = (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2$$

where $\alpha_1, \alpha_2, \alpha_3$ are the roots of $f(x)$. Also recall from HW1 that if $f(x) = x^3 + ax + b$, then the discriminant

$$\Delta = -4a^3 - 27b^2.$$

Show that $\mathrm{Gal}(L/Q) \cong \mathbb{Z}_3$ if Δ is a square of a rational number and is S_3 otherwise.

(c) Does there exist a cubic polynomial $f(x) \in \mathbb{Q}[x]$ with three real roots such that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.

Problem 10.2. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree p where p is prime. Let $\mathbb{Q} \subseteq L$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Show that if $f(x)$ has precisely $p-2$ real roots, then $\mathrm{Gal}(L/Q) \cong S_p$.

Hint: Use the lemma proved in class regarding when subgroups of S_p are the entire group.

Problem 10.3.

(1) Show that the polynomial $x^5 - 4x^2 + 2 \in \mathbb{Q}[x]$ is not solvable by radicals.

(2) Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals.