Voisin - Schiffer variations of hypersurfaces and the generic Torelli theorem

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Generic torelli theorem for hypersurfaces via Schiffer variation

Voisin

The meaning of generic Torelli thm: $X \subset \mathbb{P}^n$ very general hypersurface & X' another hypersurface $H^{n-1}_{pr}(X,\mathbb{Q}) \underset{\text{Hodge isom}}{\overset{\varphi}{\cong}} H^{n-1}_{pr}(X',\mathbb{Q}) \Rightarrow X = X'$?

Rmk: ① Weaker than Torelli i.e. period map is only generically one to one.

② Stronger: isom is with $\mathbb{Q}$-coeff not with $\mathbb{Z}$.
(e.g. for curves Torelli is wrong with $\mathbb{Q}$).

③ Usually for period map one needs polarised HS here no polarisation needed.
except for $(d,n) = (3,3)$ polarisation of $H^{n-1}_{pr}(X,\mathbb{Q})$ is unique. So it has to preserve polarisation.

④ Consider set of $S = \{(t,t') \in U_{d,n} \times U_{d,n} \mid \exists \varphi : H^{n-1}_{pr}(X_t,\mathbb{Q}) \cong H^{n-1}_{pr}(X_{t'},\mathbb{Q})\}$
set is a Hodge locus $\Rightarrow$ Deligne-Cattani-Kaplan
$S = \bigcup_{\text{cntbl.}}$ closed subsets.

$U_{d,n}$ = universal family.

1983: Thm (Donagi): The generic Torelli thm holds if $(d,n) \neq (3,3)$ (HS doesn't vary but we have a full moduli) with possible exclusion
① $d \mid n+1$ ② $d = 4$ $n+1 \equiv 2 \pmod 4$
③ $d = 6$ $n+1 \equiv 3 \pmod 6$

Thm (Voisin): Out of Donagi's leftover cases there are only finitely many exceptions to the generic Torelli thm

Where does Donagi fail? Suppose that X v. general hypersurface, X' some other

have $H^{n-1}_{pr}(X, \mathbb{Q}) \xrightarrow[\sim]{\varphi} H^{n-1}_{pr}(X', \mathbb{Q})$ $\qquad 0 \in U_{d,n}$, $X_0 = X$
$\qquad\qquad 0' \in U_{d,n}$, $X_{0'} = X'$

Consider $\Gamma_\varphi = \left\{ (t,t') \in U_{d,n} \text{ t near 0 \& t' near 0'} \mid \varphi : H^{n-1}(X_t, \mathbb{Q})_{prim} \simeq H^{n-1}_{pr}(X_{t'}, \mathbb{Q}) \right\}$

$$\begin{tikzcd}
 & \Gamma_\varphi \arrow[dl, "\text{local holo isom}"'] \arrow[dr, "\text{local holo isom}"] & \\
U & & V
\end{tikzcd}$$

$x \in$
V.general

when $(d,n) \neq (3,3)$
period map is loc. isom.

$\Rightarrow \exists\ 0 \in U \subset U_{d,n}$ + a local hol. diffeom.
$0' \in V \subset U_{d,n}$

$\forall t \in U$ $\varphi : H^{n-1}(X_t, \mathbb{Q})_{pr} \xrightarrow{\sim} H^{n-1}(X_{t'}, \mathbb{Q})_{pr}$ isom of HS

$\rightsquigarrow \psi : U \longrightarrow V$
$\qquad t \longmapsto t'$

$$\begin{tikzcd}
T_{U_{d,n},0} \arrow[r, "\psi_*"] \arrow[d, "dp"'] & T_{U_{d,n},0'} \arrow[d, "dp"] \\
\mathrm{Hom}(H^{p,q}(X_0), H^{p-1,q+1}(X_0)) \arrow[r] & \mathrm{Hom}(H^{p,q}(X_{0'}), H^{p-1,q+1})
\end{tikzcd}$$

$d(\text{period}) \longrightarrow dp$

$S = \mathbb{C}[x_0, \dots, x_n]$
$X_0 = (f=0)$ $\qquad R_f^* = S^*/J_f^*$ $\qquad$ Griffith's residue:
$R_f^{(q+1)d-n-1} \longrightarrow H^{n-q-1,q}(X_f)_{pr.}$

$$\begin{tikzcd}
T_{U_{d,n},0} \arrow[r, "dp"] \arrow[d, "\wr"'] & \bigoplus \mathrm{Hom}(H^{p,q}_{pr}, H^{p-1,q+1}_{pr}) \\
R_f^d \arrow[r, "\text{Kod-Spencer}"] \arrow[ur, phantom, "\text{commutes upto scalar } (-1)"] & \bigoplus_q \mathrm{Hom}(R^{(q+1)d-n-1}, R^{(q+2)d-n-1})
\end{tikzcd}$$

(*)

except in the exceptional dim or deg f can be reconstructed from (*).

More precisely, the trick of Donagi is that in (*) we don't have full Jacobian ring but he says (*) determines the whole Jacobian ring. Therefore $R_f^* \simeq R_{f'}^*$. $\quad X' = (f'=0)$
$\qquad X = (f=0)$

$\Longrightarrow$ $f \simeq f'$ under $GL(n+1)$
$R_f^1 = S^1$ $\qquad \left[R_f^* \rightsquigarrow J_f^d \hookrightarrow S^d \twoheadrightarrow R_f^d \right]$

Set $0 \to J_f^d \to S^d \to R_f^d \to 0$

$$0 \to J_{f'}^d \to S^d \to R_{f'}^d \to 0$$

Mather-Yau: If you have two hypersurfaces with $J_f \cong J_{f'}$ then $f \cong f'$ $(GL(n+1))$. If f is generic $f = \lambda f'$ $\lambda \in \mathbb{C}$.

$\boxed{//}$

Why $d|n+1$ is a problem?

Donagi uses symmetrizer lemma $\rightsquigarrow$ allows to reconstr. mult. in R_f^* from

$$R_f^d \to \mathrm{Hom}(R_f^{(q+1)d-n-1}, R_f^{(q+2)d-n-1})$$

when $d|n+1$,

symmetrizer lemma doesn't bring new info

Hodge theory (IVHS of X_f) gives R_f^{*d}.

Main new idea: Schiffer variations $f \in S^d$.

is a one param family $f + tx^d$

with $x \in S^1$ a linear form.

Why schiffer? For curve C its a deformation of C supp at a pt p.

i.e. $\mathbb{P}(H^1(C, T_C)) = \mathbb{P}(H^0(C, 2K_C)^\vee)$

$$\cup$$

$$[H_p = H^0(C, 2K_C - p)]$$

These are schiffer

Note, deform. are supp at p.

First order schiffer variation: all $x^d \in R_f^d$

Lemma: f is generic $d \gg 0$. Then the set of first order Schiffer variation determines f.

proof: First order:

$$S^d \to R_f^d \quad \text{induces}$$
$$x^d \mapsto x^d$$

$$\begin{array}{ccc} \mathbb{P}(R_f^d) & \dashleftarrow & \mathbb{P}(S^d) \\ \cup & & \cup \\ \text{incomplete Veronese} & \longleftarrow & \text{Veronese} \end{array}$$

Get f by Mather–Yau thm.

$L_W = i^*\mathcal{O}(1)$ $\mathcal{O}(1)$

Defn: complete embedding : $W \overset{i}{\hookrightarrow} \mathbb{P}^n$.

$$H^0(\mathbb{P}^n, \mathcal{O}(1)) \twoheadrightarrow H^0(W, L_W)$$

~~Proof of~~ main thm: Characterise schiffer variation via Hodge theory.

Part (1): $\varphi = x^d \in R_f^d$. R_f^{d*} artinian ring

Say $I_k^{*d} \subset R_f^{*d}$

I_k^{*d} = ideal generated by x^k.

$\dim I_k^{*d} = \dim R_f^{*d-k}$ for $* \leq 3$, $k \leq d$

have, $I_k^{\,d} \cdot I_{d-k}^{d} \subset \varphi R_f^d$.

In part. $I_{d-1} = \langle x^{d-1} \rangle$

Part (2)

Lemma: Along schiffer variation $R_{f_t}^{*d} / I_{d-1}^{*d}$ is const

proof: $f_t = f + tx^d \implies \dfrac{\partial f_t}{\partial x_i} = \dfrac{\partial f}{\partial x_i} \pmod{x^{d-1}}$ $\square$

+ Deform of IVHS. (second order)

Propn: If f is generic & $d \gg 0$ then (1) + (2) characterise schiffer variation

Qn Where is $d \gg 0$ needed.

Ans. $x_i^d : R_f^d \longrightarrow R_f^{2d}$ I want this to be injective for f generic. (need this for Part (1)).

e.g. This needs $d > \frac{(n+1)(d+1)}{3}$ when

$f =$ Fermat polynomial $= \sum x_i^d$

$$R_f^* = \mathbb{C}[x_1, \ldots, x_n] / (x_i^{d-1}) = H^{2*}((\mathbb{P}^{d-2})^{n+1})$$

Qn. Counterexample to Torelli for hypersurface.

Ans. Not that's known. No idea.