Semi-Ample Asymptotic Syzygies

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ASYMPTOTIC SYZYGIES

- Asymptotic syzygies is the study of the algebraic Betti numbers of a variety as the positivity of the embedding increases.

$$X \hookrightarrow \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}$$

n dimensional smooth projective variety

sequence of very ample line bundles

ASYMPTOTIC SYZYGIES

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- To this we associate:

$$S(X, L_d) = \bigoplus_{k \in \mathbb{Z}} H^0(X, kL_d)$$

- which we think of as a module over

$$S = \operatorname{Sym} H^0(L_d) \cong \mathbb{C}\left[x_0, x_1, \ldots, x_{r_d}\right]$$

ASYMPTOTIC SYZYGIES

$$0 \longleftarrow S(X, L_d) \longleftarrow \left[F_0 \longleftarrow F_1 \longleftarrow \cdots \qquad \cdots \longleftarrow F_{r_d} \right] \longleftarrow 0$$

minimal graded free resolution

$$\beta_{p,q}(X, L_d) = \# \left\{ \begin{matrix} \text{minimal generators} \\ \text{of } F_p \text{ of degree } q \end{matrix} \right\} = \begin{matrix} \text{number of syzygies of degree } q \\ \text{and homological degree } p \end{matrix}$$

how do these vary as a function of d?

ASYMPTOTIC SYZYGIES

- It is often useful to place the Betti numbers into a table:

$$\beta(X,d) = \begin{array}{c|ccccccc} & 0 & 1 & 2 & \cdots & p & \cdots \\ \hline 0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots & \cdots & \\ 1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \\ q & & & & & \beta_{p,p+q} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

notice the wonky change of coordinates

FIRST RESULTS - CURVES

$$X \hookrightarrow \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}$$

smooth projective curve of genus g

line bundle of degree d

Theorem (Castelnuovo et al.).

1. If $d \geq 2g+1$ then L_d is defines an embedding into $\mathbb{P}^{r_d}$.
2. If $d \geq 2g+2$ then I_X is generated by quadrics.

FIRST RESULTS - CURVES

- Part (1) tells us the Betti table of X eventually looks like:

	0	1	2	$\cdots$	p	$\cdots$	r_d
0	1	-	-	$\cdots$	-	$\cdots$	
1	-	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{p,p+1}$	$\cdots$	
2	-	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$	$\beta_{p,p+2}$	$\cdots$	

correspond to the generators
of the defining ideal

- Part (2) tells us the Betti table of X eventually looks like:

	0	1	2	$\cdots$	p	$\cdots$	r_d
0	1	-	-	$\cdots$	-	$\cdots$	
1	-	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{p,p+1}$	$\cdots$	
2	-	-	$\beta_{2,4}$	$\cdots$	$\beta_{p,p+2}$	$\cdots$	

FIRST RESULTS - CURVES

$$\rho_q(X;L_d) \;:=\; \frac{\#\left\{\, p \in \mathbb{N} \;\middle|\; \beta_{p,p+q}(X,L_d) \neq 0 \,\right\}}{r_d}$$

the percentage
of non-zero syzygies
in row q

Theorem (Green, 1984). Let X be a smooth curve. If $\deg L_d = d$ then

$$\lim_{d\to\infty} \rho_2(X;L_d) \;=\; 0.$$

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as the degree of
the line bundle grows

the percentage
of non-zero syzygies
in the second row

FIRST RESULTS - AMPLE GROWTH

Theorem (Ein-Lazarsfeld, 2012). Let $n \geq 2$ and fix $1 \leq q \leq n$. If $L_{d+1} - L_d$ is constant and ample then

$$\lim_{d\to\infty} \rho_q(X; L_d) = 1$$

the percentage of non-zero syzygies

asymptotically syzygies occur in every possible degree

FIRST RESULTS - AMPLE GROWTH

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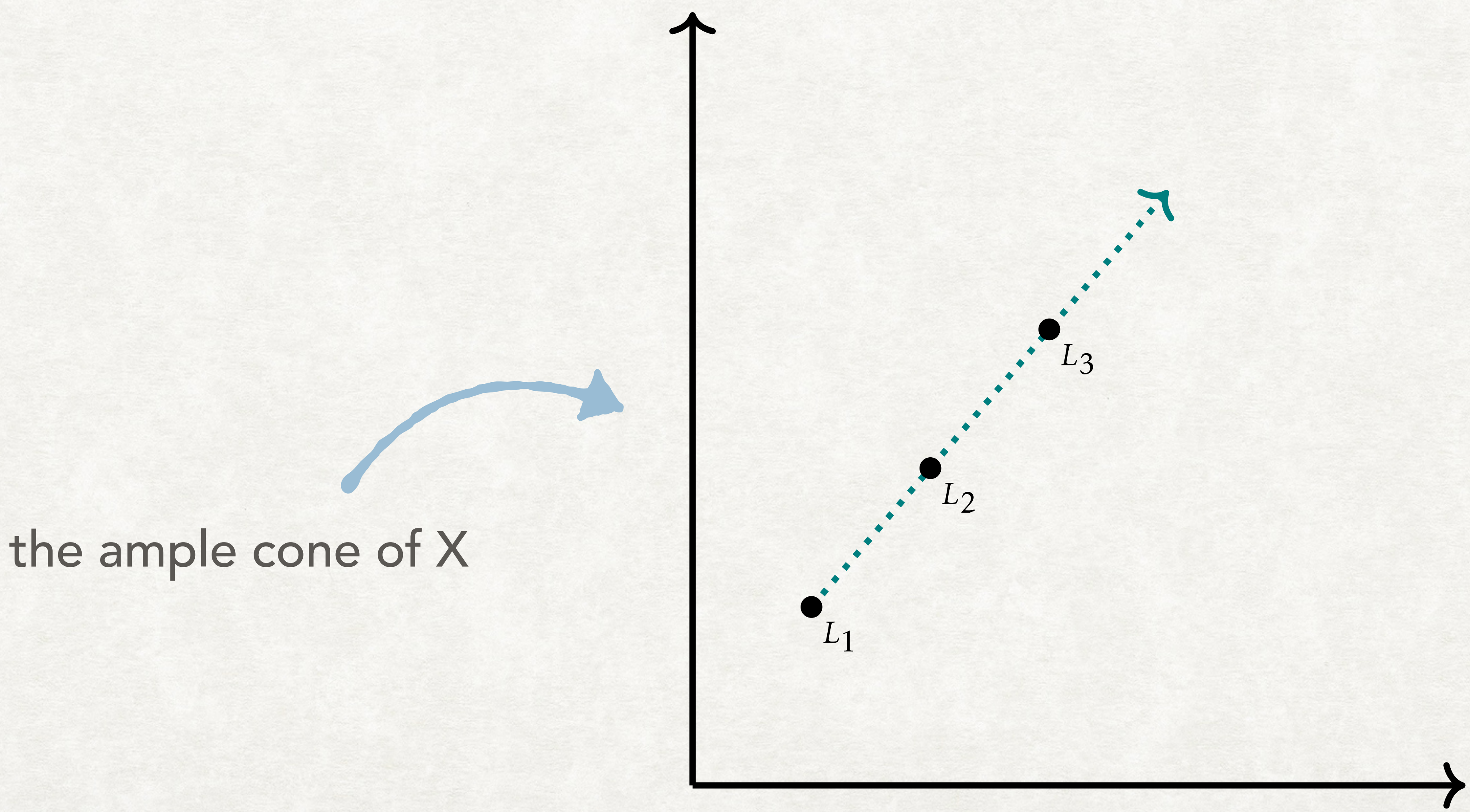


NEW RESULTS - SEMI-AMPLE GROWTH

Theorem (Juliette Bruce)**.** Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exists constants $C_{i,j}$ and $D_{i,j}$ such that

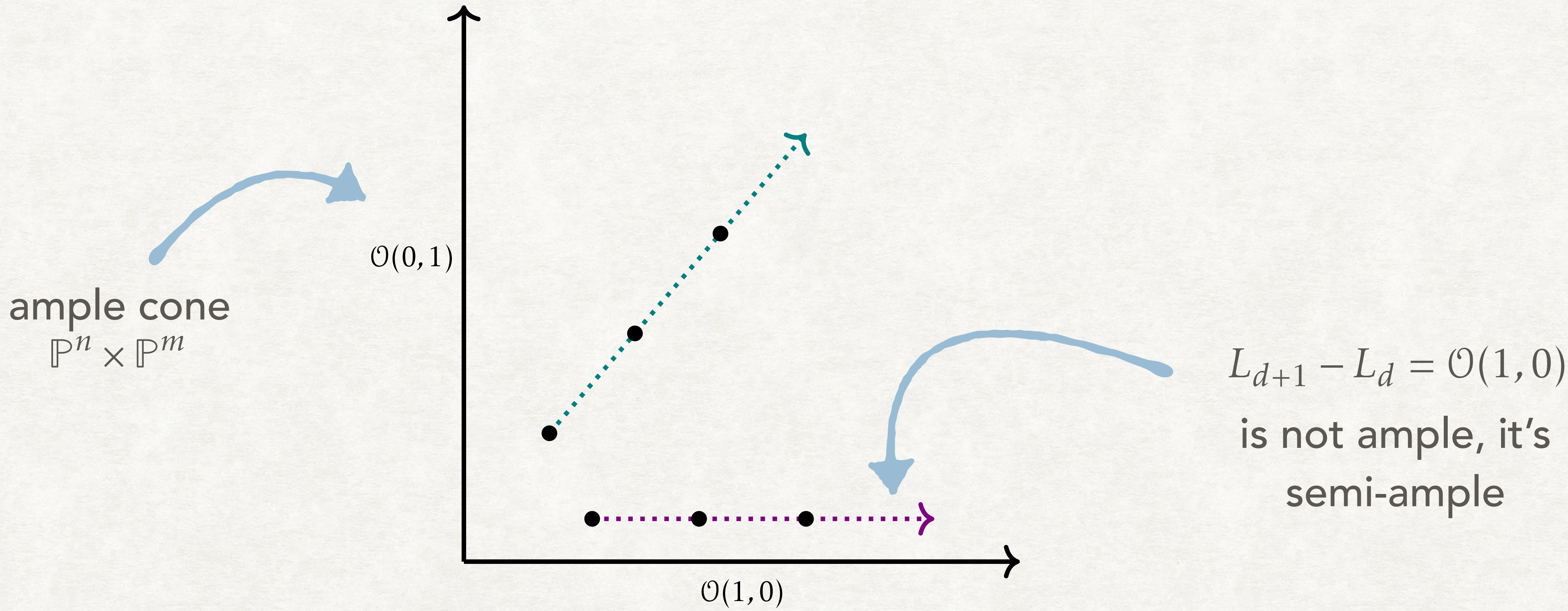
$$\rho_q(X;\mathcal{O}(d_1,d_2)) \geq 1 - \sum_{\substack{i+j=q \\ 0\leq i\leq n \\ 0\leq j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}} \right) - O\begin{pmatrix}\text{lower ord.} \\ \text{terms}\end{pmatrix}.$$

the percent of possible degrees with non-zero syzygies

the main specific asymptotic behavior

NEW RESULTS - SEMI-AMPLE GROWTH

- My result does not require an assumption of ample growth.



Definition. A line bundle L is semi-ample if $|kL|$ is base point free for some k.

NEW RESULTS - SEMI-AMPLE GROWTH

- If n = 1 and m = 5 then my result shows:

$$\rho_2\left(\mathbb{P}^1 \times \mathbb{P}^5; \mathcal{O}(d_1, d_2)\right) \geq 1 - \frac{20}{d_2^2} - \frac{60}{d_1 d_2^3} - \frac{5}{d_1 d_2} - \frac{120}{d_2^4} - O\begin{pmatrix}\text{lower ord.} \\ \text{terms}\end{pmatrix}$$

- In particular, if d_2 is fixed then

$$\lim_{d_1 \to \infty} \rho_2\left(\mathbb{P}^1 \times \mathbb{P}^5; \mathcal{O}(d_1, d_2)\right) \geq 1 - \frac{20}{d_2^2} - \frac{120}{d_2^4}.$$

neither 0 nor 1

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- Syzygies in the setting of semi-ample growth exhibit behavior that is different from previous cases!

  - curves (Green)
  - ample growth (Ein-Lazarsfeld)

APPROACH OF PROOF

- The proof generalizes the monomial methods of Ein, Erman, Lazarsfeld to explicitly produce non-trivial syzygies.

this requires an Artinian reduction, but there are no monomial regular sequences

- If n = 2 and m = 4 the regular sequence we work with is:

$$\left\langle \begin{matrix} x_0^{d_1} y_0^{d_2} \\ x_0^{d_1} y_1^{d_2} + x_1^{d_1} y_0^{d_2} \\ x_0^{d_1} y_2^{d_2} + x_1^{d_1} y_1^{d_2} + x_2^{d_1} y_0^{d_2} \\ x_0^{d_1} y_3^{d_2} + x_1^{d_1} y_2^{d_2} + x_2^{d_1} y_1^{d_2} \\ x_0^{d_1} y_4^{d_2} + x_1^{d_1} y_3^{d_2} + x_2^{d_1} y_2^{d_2} \\ x_1^{d_1} y_4^{d_2} + x_2^{d_1} y_3^{d_2} \\ x_2^{d_1} y_4^{d_2} \end{matrix} \right\rangle$$

$x_0^{d_1-1} x_1^{d_1-1} x_2^{3d_1+2} y_0^{3d_2-1} y_1^{d_2-1} y_2^{d_2-1} y_3^3$

is not in this ideal, while

$x_0^{d_1-1} x_1^{d_1-1} x_2^{3d_1+2} y_0^{3d_2} y_1^{d_2-1} y_2^{d_2-1} y_3^3$

is in this ideal…

APPROACH OF PROOF

- I exploit the fact that this regular sequence has a lot of symmetries:

i.e. homogenous with respect to a grading

the sum of the indices is constant

$$\left\langle \begin{matrix} x_0^{d_1} y_0^{d_2} \\ x_0^{d_1} y_1^{d_2} + x_1^{d_1} y_0^{d_2} \\ x_0^{d_1} y_2^{d_2} + x_1^{d_1} y_1^{d_2} + x_2^{d_1} y_0^{d_2} \\ x_0^{d_1} y_3^{d_2} + x_1^{d_1} y_2^{d_2} + x_2^{d_1} y_1^{d_2} \\ x_0^{d_1} y_4^{d_2} + x_1^{d_1} y_3^{d_2} + x_2^{d_1} y_2^{d_2} \\ x_1^{d_1} y_4^{d_2} + x_2^{d_1} y_3^{d_2} \\ x_2^{d_1} y_4^{d_2} \end{matrix} \right\rangle$$

the power on each x is 0 mod d_1

- These symmetries enable me to use spectral sequence arguments to deeply understand this ideal.