Affineness of the complement of the ramification locus

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April 19, 2020

Schemes are separated, Noetherian, excellent

X normal integral scheme

Y regular integral scheme

$f : X \to Y$ is a dominant finite type morphism

$\dim(X/Y) = \dim(X_\eta)$ where $\eta \in Y$ generic point

$\mathrm{Sing}(f)$ locus in X where f isn't smooth

Purity Question: Is

$\mathrm{codim}(\mathrm{Sing}(f), X) \leq 1 + \dim(X/Y)$?

Yes for relative dimension 0:

1. Zariski-Nagata purity of ramification locus,
2. van der Waerden purity for birational maps,
3. general case by Gabber in a paper of Zong (2014)

Discussion in EGA IV Section 21.12 suggests

Theorem In equicharacteristic if $\dim(X/Y) = 0$ the embedding $V = X \setminus \text{Sing}(f) \to X$ is affine.

Implies all 3 cases when it applies. Open in mixed char as far as I know.

Let me sketch a proof.

By induction on the dimension we reduce to:
$(X, x) \to (Y, y)$ local, dim$(Y) \geq 2$
$x \in$ Sing(f),
$V \to X \setminus \{x\}$ affine, and
$V \to Y$ étale

Lemma $H^i(V, \mathcal{O}_V)$, $i > 0$ is a direct sum of copies of the injective hull E of the residue field $\kappa(y)$ of $\mathcal{O}_{Y,y}$

Proof The étaleness of $V \to Y$ shows that differential operators on $\mathcal{O}_{Y,y}$ act on $H^i(V, \mathcal{O}_V)$ for all i (in positive characteristic one uses Frobenius). Also $H^i(V, \mathcal{O}_V)$ for $i > 0$ is $\mathfrak{m}_y$-power torsion (cohomology already zero away from y). Apply a standard structure theorem about torsion modules over $k[[x_1, \ldots, x_n]]$ endowed with actions of all differential operators (or a suitable Frobenius structure).

Lemma $H^{\dim(Y)-1}(V, \mathcal{O}_V) = 0 \Rightarrow V$ affine

Proof Since y is not in the image of $V \to Y$ we have $R\Gamma(V, \mathcal{O}_V) \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,y}} \kappa(y) = 0$. Computing Tors of E we conclude $\mathfrak{m}_y \cdot \Gamma(V, \mathcal{O}_V) = \Gamma(V, \mathcal{O}_V)$. Then V is affine because we already know that $V \times_Y D(a)$ is affine for $a \in \mathfrak{m}_y$ (lemma in Hartshorne).

End of the proof of theorem By dimension formula $\dim(X) \leq \dim(Y)$. By purity (due to Gabber) we know that V is not all of $X \setminus \{x\}$ and hence we have vanishing of $H^{\dim(Y)-1}(V, \mathcal{O}_V)$ by Hartshorne-Lichtenbaum vanishing.

Results for relative dimension > 0:

Warning: I am not sure that the answer to the question should be yes!

Dolgachev proved the answer is yes in case $f : X \to Y$ is a local complete intersection morphism. In particular, if $f : X \to Y$ is a morphism over S and X, Y are smooth over S, then this is true. This can be found in a paper of Rolf Källström.

Besides the result of Dolgachev I can prove a subcase of the relative dimension 1 case, as I will explain on the next slide.

Special case in relative dimension 1:
$(X, x) \to (Y, y)$ local, $\dim(Y) = 2$
$\mathrm{Sing}(f) = \{x\}$,
$X \setminus \{x\} \to Y$ smooth of relative dimension 1
We have to show this doesn't happen.

Idea: Given $f : X \to Y$ as above we can produce a proper flat family of curves $X' \to Y$ and a point $x' \in X'$ such that the completion of X' at x' is isomorphic to the completion of X at x as formal Y-schemes and such that $f' : X' \to Y$ is smooth away from x'. Then purity for families of smooth proper curves (proved by Moret-Bailly) shows that $X' \setminus (f')^{-1}(\{y\})$ can be extended to a smooth (!) proper family $X'' \to Y$ of curves over Y. Then you show that $X' \cong X''$ over Y and you conclude that x' wasn't a singular point of the fibre to begin with.

Thank you.