Some vignettes on sums-of-squares on varieties.

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Main characters: Two convex cones P and Σ

Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be a multivariate polynomial with real coefficients.

Definition.

The polynomial f is **nonnegative** *($f \in P$) if $f(\alpha) \geq 0$ for every $\alpha \in \mathbb{R}^n$.*

Definition.

The polynomial f is a **sum-of-squares** *($f \in \Sigma$) if there exist an integer $t > 0$ and polynomials $g_1, \ldots, g_t \in \mathbb{R}[X_1, \ldots, X_n]$ such that*

$$f = g_1^2 + \cdots + g_t^2.$$

Nonnegative polynomials (P)

The cone of nonnegative polynomials is important because *it allows us formulate global optimization problems*:

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Such reformulations have many applications (see for instance J.B. Lasserre's "Moments, positive polynomials and their applications")

Sums-of-squares (Σ)

Sums of squares provide certificates of nonnegativity:

Example:

Is the following polynomial f nonnegative in $\mathbb{R}^2$?

$$f = 10x^6 - 4x^5y + 2x^4y^2 + 50x^4 - 14x^3y - 4x^3 + 4x^2y + 65x^2 - 14x + 2$$

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Is the following polynomial f nonnegative in $\mathbb{R}^2$?
$f = (1 + x + x^3 + x^2y)^2 + (1 - 8x - 3x^3 + x^2y)^2.$

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Remark.

A polynomial f is a sum-of-squares of elements of V if and only if there exists a symmetric matrix $A \in \mathbb{R}^{e \times e}$ such that

$$A \succeq 0 \quad \text{and} \quad f = \vec{m}^t A \vec{m}$$

where $\vec{m} = (h_1, \ldots, h_e)^t$ is a vector whose entries are a basis for V.

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Constructing SOS certificates reduces to semidefinite programming feasibility.

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For which degrees $2d$ and number of variables n is every nonnegative form (homogeneous polynomial) of degree $2d$ a sum-of-squares?

Theorem. (Hilbert 1888)

Every nonnegative form of degree $2d$ in in n-variables is a sum-of-squares if and only if either,

1. $n = 2$ *(bivariate forms) or*
2. $d = 1$ *(quadratic forms) or*
3. $n = 3$ *and* $d = 2$ *(ternary quartics).*

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Question.

Can we find a natural context where we can **understand** *and hopefully* **generalize** *Hilbert's Theorem?*

Real projective varieties

Let $X \subseteq \mathbb{P}^n$ be a real projective variety (reduced, not necessarily irreducible) and let $S := \mathbb{R}[X_0, \dots, X_n]/I(X)$ be its homogeneous coordinate ring.

Definition.

The cone of nonnegative quadratic forms P_X is given by

$$P_X = \{f \in S_2 : \forall \alpha \in X(\mathbb{R})\,(f(\alpha) \geq 0)\}$$

Definition.

The cone of sums-of-squares of linear forms

$$\Sigma_X = \left\{f \in S_2 : \exists s_1, \dots, s_t \in S_1 : f = \sum s_i^2\right\}$$

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In principle, restricting only to quadratic forms seems to be fairly restrictive. However, this is not the case since we are considering arbitrary varieties so quadratic forms in $\nu_d(X)$ correspond to $2d$-forms on X.

A partial answer: irreducible varieties.

Let $X \subseteq \mathbb{P}^n$ be a real projective variety. Assume:

1. X is non-degenerate and totally real.
2. X is **irreducible**.

Theorem. (Blekherman, Smith, - , 2016)

The equality $P_X = \Sigma_X$ occurs if and only X is a variety of minimal degree (i.e. if the equality $\deg(X) = 1 + \mathrm{codim}(X)$ *holds).*

Varieties of minimal degree

If $X \subseteq \mathbb{P}^n$ is a positive-dimensional, irreducible and non-degenerate variety then its general hyperplane section is non-degenerate. It follows that

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Theorem. (Del Pezzo, Bertini, 1880)

Let $X \subseteq \mathbb{P}^n$ be irreducible and not contained in any hyperplane in $\mathbb{P}^n$. If X is of minimal degree (i.e. $\deg(X) = \mathrm{codim}(X) + 1$) then either:

1. *$X = \mathbb{P}^n$ or*
2. *X is a quadric hypersurface or*
3. *X is a cone over the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or*
4. *X is a rational normal scroll.*

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1. $P_X = \Sigma_X$ is preserved under projections away from real points,
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1. $P_X = \Sigma_X$ is preserved under projections away from real points,
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   2. For hypersurfaces $P_X = \Sigma_X$ iff X is a quadric hypersurface.
2. $P_X \neq \Sigma_X$ is preserved under generic hyperplane sections (By our *Bertini-type theorem for separators* convex geometry + complex geometry).
   1. Slice X with a complementary subspace to obtain a set of points with $P_X \neq \Sigma_X$.
   2. For a set of points X equality holds iff X is a linearly independent set.

Consequences

We could **unify** and **generalize** results scattered in the literature:

1. $X = \nu_d(\mathbb{P}^n)$ is minimal degree if and only if... (Hilbert's Theorem 1888).
2. $X = V(Q)$... (Yakubovich's Theorem 1971)
3. $X = \sigma_{d_1,d_2}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2})$ is minimal degree if and only if... (Choi-Lam-Reznick 1980)
4. New SOS results on nonnegative polynomials with special support from rational normal scrolls (2016).

Vignette 1: How about denominators?

In 1927 Artin showed (solving Hilbert 17th) that every nonnegative polynomial admits a representation as a sum-of-squares of rational functions (and in particular as a ratio of sums of-squares).

Given $f \in P$ *find* $g \in \Sigma : fg \in \Sigma$.

Question.

Do such representations exist on varieties?

Theorem. (Blekherman, Smith, -, 2019)

Let $X \subseteq \mathbb{P}^n$ *be a totally real, non-degenerate curve of degree* d *and arithmetic genus* p_a. *If* $f \in P_{X,2j}$ *and* $k \geq \frac{2p_a}{d}$ *then there exists* $g \in \Sigma_{X,2k}$ *such that* $fg \in \Sigma_{X,2(j+k)}$. *These bounds are* **sharp**.

Vignette 2: Efficiency of representations

In 1984 Pfister showed that every nonnegative form in $\mathbb{R}^n$ has a rational SOS representation involving at most 2^n squares.

Definition.

The **pythagoras number** $\Pi(X)$ *of a projective variety* $X \subseteq \mathbb{P}^n$ *is the smallest number of squares that suffices to write ANY element of* Σ_X.

Theorem. (Blekherman, Smith, Sinn, -, 2020)

If X *is totally real, irreducible, non-degenerate and arithmetically Cohen-Macaulay then the following conditions are equivalent:*

1. $\Pi(X) = 2 + \dim(X)$ *(next-to-minimal)*
2. $\deg(X) = 2 + \operatorname{codim}(X)$ *or* X *is codimension one in a variety of minimal degree.*

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