Schur-Positive Symmetric Functions From Polynomial Products

Isaiah Siegl
University of Washington

AMS Southwest Sectional Fall 2026

11/07/2026

Symmetric Function Bases

- A powerseries $f(\mathbf{x})$ is *symmetric* if $f(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots)$ for all $i \in \mathbb{Z}_{>0}$.
- Monomial symmetric functions

$$m_\lambda(\mathbf{x}) = \sum_{\substack{\alpha \vDash n \\ \mathrm{sort}(\alpha)=\lambda}} \prod_{i \geq 1} x_i^{\alpha_i}.$$

- Elementary symmetric functions

$$e_k(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad e_\lambda(\mathbf{x}) = e_{\lambda_1}(\mathbf{x}) e_{\lambda_2}(\mathbf{x}) \cdots e_{\lambda_l}(\mathbf{x}).$$

- Schur symmetric functions

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} x_T$$

Real Rooted Polynomials

Theorem (Stanley '98)

Let $F(t) = 1 + a_1 t + \cdots + a_d t^d$ be a polynomial with real coefficients. The following are equivalent.

- $\Omega_F(\mathbf{x}) = \prod_{i \geq 1} F(x_i)$ *is Schur-positive.*
- $\Omega_F(\mathbf{x}) = \prod_{i \geq 1} F(x_i)$ *is e-positive.*
- *All roots of $F(t)$ are negative real numbers.*
- *The matrix $[a_{j-i}]_{i,j=1}^{\infty}$ is totally non-negative.*

Question

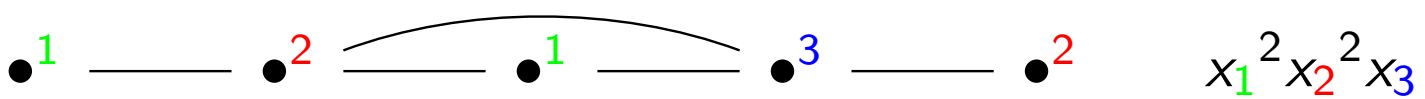
What if $F(t)$ has coefficients in $\mathbb{R}[z_1, z_2, \ldots, z_n]$? Other coefficient rings?

Chromatic Symmetric Functions

Definition (Stanley '95)

For a graph $G = (V, E)$ define the chromatic symmetric function $X_G(\mathbf{x})$ by

$$X_G(\mathbf{x}) = \sum_{\substack{\kappa: V \to \mathbb{Z}_{>0} \\ \kappa(v) \neq \kappa(w) \text{ if } \{v,w\} \in E}} \prod_{v \in V} x_{\kappa(v)}.$$

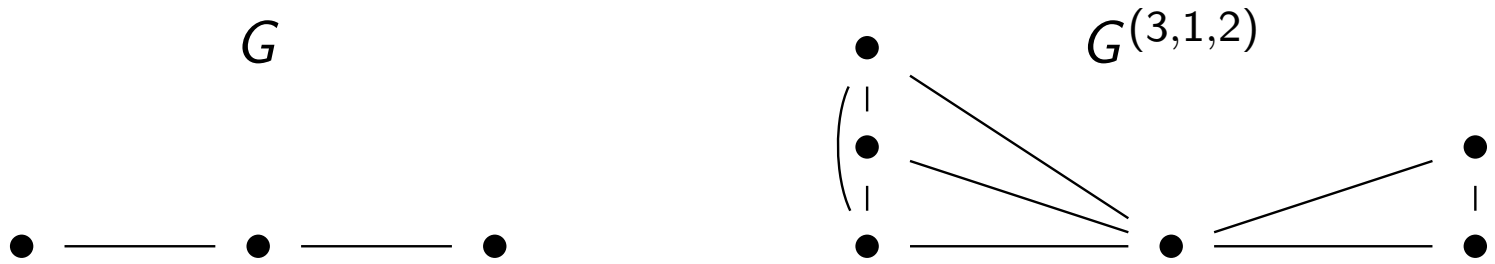


$$x_1^2 x_2^2 x_3$$

Multivariate Independence Polynomial

Definition

Define the multivariate independence polynomial $I_G(\mathbf{z}, t)$ *of a graph* G *by* $I_G(\mathbf{z}, t) = \sum_S t^{|S|} \prod_{v \in S} z_v$, *where the sum over all independent sets of* G.



Theorem (Stanley '98)

$$\prod_{i \geq 1} I_G(\mathbf{z}, x_i) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{\mathbf{z}^\alpha}{\alpha!} X_{G^\alpha}(\mathbf{x}).$$

Dual Cauchy Product

For $\mathbf{u} = (u_1, u_2, \ldots)$, the *dual Cauchy product* is the identity

$$\Omega(\mathbf{u}, \mathbf{x}) = \prod_{i,j \geq 1} (1 + u_i x_j) = \prod_{j \geq 1} \sum_{k \geq 0} e_k(\mathbf{u}) x_j^k$$
$$= \sum_{\lambda} e_\lambda(\mathbf{u}) m_\lambda(\mathbf{x}) = \sum_{\lambda} s_\lambda(\mathbf{u}) s_{\lambda'}(\mathbf{x}) = \sum_{\lambda} m_\lambda(\mathbf{u}) e_\lambda(\mathbf{x}).$$

- Let $F(t) = 1 + a_1 t + \cdots + a_d t^d = \prod(1 + \alpha_i t)$. Then
$$\prod_{i \geq 1} F(x_i) = \Omega(\alpha, \mathbf{x}).$$
- If $\alpha \in \mathbb{R}^d_{\geq 0}$, then $\prod_{i \geq 1} F(x_i)$ is Schur-positive and e-positive.
- For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $s_{\lambda'}(\mathbf{u}) = \Delta_{I,J}[e_{j-i}(\mathbf{u})]_{i,j=1}^{\infty}$, where $I = \{1, 2, \ldots, \ell\}$ and $J = \{\lambda_\ell, \lambda_{\ell-1} + 1, \ldots, \lambda_1 - \ell + 1\}$.

F-Analogues

- For a polynomial $F(\mathbf{z}, t) \in \mathbb{R}[z_1, z_2, \ldots, z_n][t]$, write $F(\mathbf{z}, t) = 1 + e_1^F(\mathbf{z})t + e_2^F(\mathbf{z})t^2 + \cdots + e_d^F(\mathbf{z})t^d$.
- For a symmetric function $g(\mathbf{x}) = p(e_1, e_2, \ldots)$, define the F-analogue $g^F(\mathbf{z}) = p(e_1^F, e_2^F, \ldots, e_d^F)$.

$$\Omega_F(\mathbf{z}, \mathbf{x}) := \prod_{i \geq 1} F(\mathbf{z}, x_i) = \prod_{i \geq 1} \sum_{k \geq 0} e_k^F(\mathbf{z}) x_i^k$$
$$= \sum_{\lambda} e_\lambda^F(\mathbf{z}) m_\lambda(\mathbf{x}) = \sum_{\lambda} s_\lambda^F(\mathbf{z}) s_{\lambda'}(\mathbf{x}) = \sum_{\lambda} m_\lambda^F(\mathbf{z}) e_\lambda(\mathbf{x}).$$

- If $\lambda = (\lambda_1, ..., \lambda_\ell)$, then $s_{\lambda'}^F(\mathbf{z}) = \Delta_{I,J}[e_{j-i}^F(\mathbf{z})]_{i,j=1}^{\infty}$, where $I = \{1, 2, \ldots, \ell\}$ and $J = \{\lambda_\ell, \lambda_{\ell-1} + 1, \ldots, \lambda_1 + \ell - 1\}$.

Symmetric Function Evaluations

- For $\mathbf{r} \in \mathbb{R}^n$, write $F(\mathbf{r}, t) = \prod_{i=1}^{d}(1 + u_i^{\mathbf{r}} t)$. Then for any symmetric function $g(\mathbf{x})$, we have $g^F(\mathbf{r}) = g(\mathbf{u}^{\mathbf{r}})$.
- Suppose $\prod_{i \geq 1} F(\mathbf{z}, x_i)$ is Schur-positive. (e-positive)
  - If $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$, then $\mathbf{u}^{\mathbf{r}} \in \mathbb{R}^d_{\geq 0}$.
  - $s_\lambda^F(\mathbf{z}) \in \mathbb{R}_{\geq 0}[\mathbf{z}]$ for all λ. $\qquad (m_\lambda^F(\mathbf{z}) \in \mathbb{R}_{\geq 0}[\mathbf{z}])$.
  - If $\mathbf{r} \geq_{entry} \mathbf{s} \in \mathbb{R}^n_{\geq 0}$, then for all λ,

$$s_\lambda(\mathbf{u}^{\mathbf{r}}) = s_\lambda^F(\mathbf{r}) \geq s_\lambda^F(\mathbf{s}) = s_\lambda(\mathbf{u}^{\mathbf{s}})$$
$$(m_\lambda(\mathbf{u}^{\mathbf{r}}) = m_\lambda^F(\mathbf{r}) \geq m_\lambda^F(\mathbf{s}) = m_\lambda(\mathbf{u}^{\mathbf{s}}).)$$

- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_{\geq 0}$, under what conditions will $s_\lambda(\mathbf{u}) \geq s_\lambda(\mathbf{v})$ for all λ?

Evaluation Inequalities

- For $\mathbf{u} \in \mathbb{R}^d_{\geq 0}$, write $u_{[i]}$ for the i^{th} largest entry of $\mathbf{u}$.
- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_{\geq 0}$, we say $\mathbf{u}$ *log-majorizes* $\mathbf{v}$ and write $\mathbf{u} \succeq_{\log} \mathbf{v}$ if for all $1 \leq k \leq d$,

$$\prod_{i=1}^{k} u_{[i]} \geq \prod_{i=1}^{k} v_{[i]}.$$

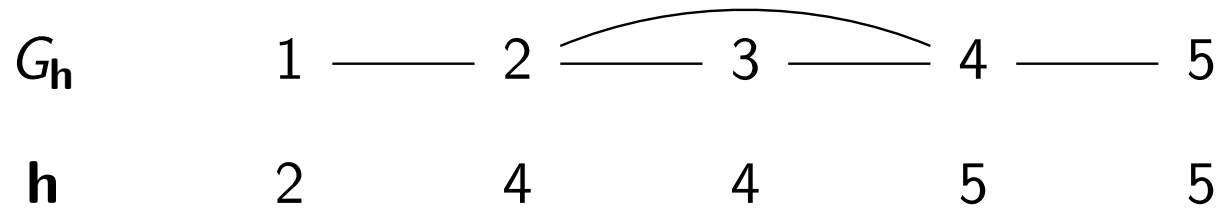
Theorem (S.)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_{\geq 0}$. The following are equivalent.

- $\mathbf{u} \succeq_{\log} \mathbf{v}$.
- $s_\lambda(\mathbf{u}) \geq s_\lambda(\mathbf{v})$ *for all partitions* λ.
- $m_\lambda(\mathbf{u}) \geq m_\lambda(\mathbf{v})$ *for all partitions* λ.

Hessenberg Functions

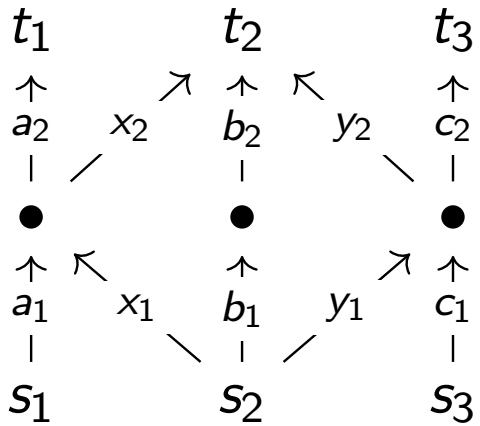
- A *Hessenberg function* is map $\mathbf{h} : [n] \to [n]$ such that
  - $h_i \leq h_{i+1}$ for all $1 \leq i < n$, and
  - $h_i \geq i$ for all $1 \leq i \leq n$.
- Define $G_{\mathbf{h}} = ([n], E)$ by $\{i < j\} \in E \iff h_i \geq j$.



- **Theorem**[Gasharov '96] $X_{G_{\mathbf{h}}}(\mathbf{x})$ is Schur-positive.
- **Theorem**[Hikita '24] $X_{G_{\mathbf{h}}}(\mathbf{x})$ is e-positive.
- Write $I_{G_{\mathbf{h}}}(\mathbf{z}, t) = \sum_{k=0}^{d} e_k^{\mathbf{h}}(\mathbf{z}) t^k$.
- If $\left[e_{j-i}^{\mathbf{h}}(\mathbf{z})\right]_{i,j=1}^{\infty}$ is totally non-negative, then $X_{G_{\mathbf{h}}}(\mathbf{x})$ is Schur-positive.

Lindström–Gessel–Viennot

- Directed planar networks correspond to totally nonnegative matrices
- Minors of TNN matrices correspond to non-intersecting path families



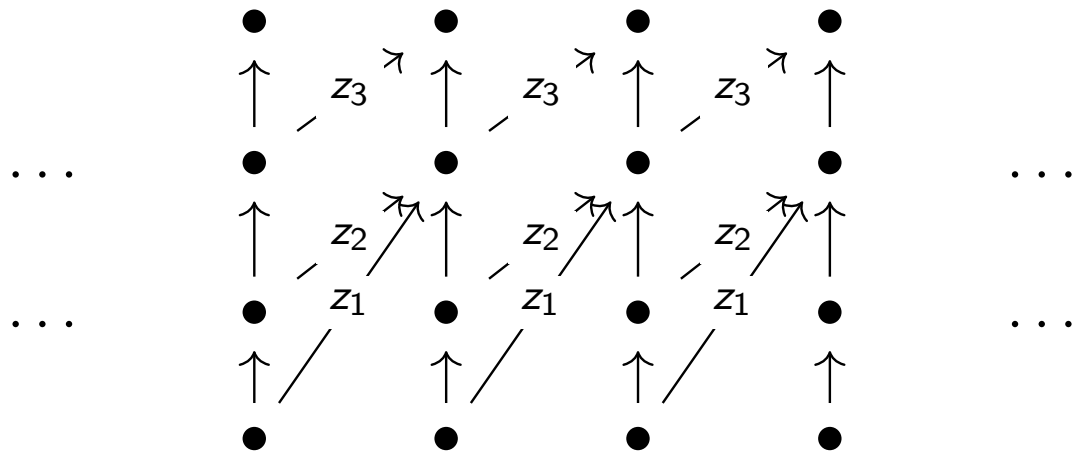
$$\leftrightarrow \begin{bmatrix} a_1a_2 & a_1x_2 & 0 \\ x_1a_2 & x_1x_2 + b_1b_2 + y_1y_2 & y_1c_2 \\ 0 & c_1y_2 & c_1c_2 \end{bmatrix}$$

- Can we find a directed planar network that proves total non-negativity of $\left[e^{\mathbf{h}}_{j-i}(\mathbf{z})\right]_{i,j=1}^{\infty}$?

$\Gamma_{\mathbf{h}}$

- Let $\mathbf{h} : [n] \to [n]$ be a Hessenberg function.
- Define $\Gamma_{\mathbf{h}}$ be the directed graph on $\mathbb{Z} \times [n+1]$ with
  - Edges of weight 1: $(i, j) \to (i, j+1)$
  - Edges of weight z_j: $(i, j) \to (i+1, h_j + 1)$.

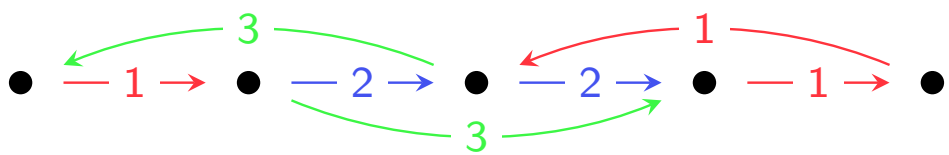
$\Gamma_{(2,2,3)}$:



- The weight matrix of $\Gamma_{\mathbf{h}}$ is $[e^{\mathbf{h}}_{j-i}(\mathbf{z})]_{i,j=-\infty}^{\infty}$.
- New proof of Schur-positivity of $X_{G_{\mathbf{h}}}(\mathbf{x})$.

Kirchhoff Symmetric Functions

- For a graph $G = (V, E)$, a *rooted forest* is a pair (A, R) such that
  - $A \subseteq E$ is a forest, and
  - $R \subseteq V$ has exactly one vertex from each connected component of A.
- Define the *Kirchhoff symmetric function* $K_G(\mathbf{x})$ of a loopless graph G by summing over all ordered partitions of G into rooted forests.



$$x_1^3 x_2^2 x_3^2$$

Theorem (S.)

For a loopless graph G, $K_G(\mathbf{x})$ is Schur-positive.

The Matrix-Tree Theorem

- For $e = \{i, j\} \in E(G)$, let $v_e = \mathbf{1}_i - \mathbf{1}_j \in \mathbb{R}^V$.
- Define the *weighted Laplacian* $\mathcal{L}_G(\mathbf{z}) = \sum_{e \in E} z_e v_e v_e^* \in \mathbb{R}[\mathbf{z}]^{V \times V}$.
- **Theorem**(Matrix-Tree) For a graph G,

$$RF_G(\mathbf{z}, t) := \sum_{(A,R) \in \mathcal{RF}(G)} t^{|A|} \prod_{e \in A} z_e = \det(I + t\mathcal{L}_G(\mathbf{z})).$$

- For $\alpha \in \mathbb{Z}_{\geq 0}^E$, let $G[\alpha]$ be the graph with α_e copies of edge e.
- **Lemma**[S.] For a loopless graph G,

$$\prod_{i \geq 1} RF_G(\mathbf{z}, x_i) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^E} \frac{\mathbf{z}^\alpha}{\alpha!} K_{G[\alpha]}(\mathbf{x}).$$

Positivity for PSD matrices

- **Theorem**[S.] Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{C}^n$, and let $A(\mathbf{z}) = \sum_{i=1}^{m} z_i \mathbf{v}_i \mathbf{v}_i^*$. Then $\prod_{i \geq 1} \det(I + x_i A(\mathbf{z}))$ is Schur-positive.
- Coefficient of $s_\lambda(\mathbf{x})$ is $s_{\lambda'}[A(\mathbf{z})]$ (evaluation at eigenvalues)
- Applying this theorem to $\mathcal{L}_G(\mathbf{z})$ gives Schur-positivity of $K_G(\mathbf{x})$.
- Do the Schur-coefficients of $K_G(\mathbf{x})$ have a nice combinatorial interpretation?
- Are there other combinatorially interesting matrices of the form $A(\mathbf{z}) = \sum_{i=1}^{n} z_i \mathbf{v}_i \mathbf{v}_i^*$?

Other Coefficient Rings

- Let $\mathcal{R}$ be a not-necessarily-commutative $\mathbb{Q}$-algebra, and let $\mathcal{R}_+ \subset \mathcal{R}$ be closed under multiplication and $\mathbb{Q}_{\geq 0}$-linear combinations.
- For $F(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_d t^d \in \mathcal{R}$, the coefficients $\{a_i\}$ must commute pairwise for $\prod_{i \geq 1} F(x_i)$ to be symmetric.
- Ex. [Fomin–Greene '98] Let $\mathcal{R} = \mathbb{Q}\langle z_1, z_2, \ldots, z_n \rangle / I$ where $I = (z_b z_a z_c - z_b z_c z_a | a < b \leq c) + (z_c z_a z_b - z_a z_c z_b | a \leq b < c)$. Then the Schur-coefficients of

$$\prod_{i \geq 1} \sum_{k \geq 0} x_i^k \sum_{i_1 > \cdots > i_k \geq 1} z_{i_1} z_{i_2} \cdots z_{i_k}$$

have positive representatives in $\mathbb{Q}\langle z_1, z_2, \ldots z_n \rangle / I$.

$\mathcal{R} = \mathbb{Q}S_n$

- For $\pi \in S_n$, $\mathrm{des}(\pi) = |\{i \in [n-1] : \pi(i) > \pi(i+1)\}|$.
- Define the n^{th} Eulerian polynomial $A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}$.
- $A_n(t)$ is real-rooted.
- **Theorem**[Loday '89] Let $A_n^{(i)} = \sum_{\mathrm{des}(\pi)=i} \pi \in \mathbb{Q}S_n$. Then $A_n^{(i)} A_n^{(j)} = A_n^{(j)} A_n^{(i)}$ for all $0 \leq i, j < n$.
- **Conjecture**[S.] Let $\mathcal{A}_n(t) = \sum_{i=1}^{n-1} A_n^{(i)} t^i = \sum_{\pi \in S_n} \pi \, t^{\mathrm{des}(\pi)}$. Then the Schur-coefficients of $\prod_{i \geq 1} \mathcal{A}_n(x_i)$ are in $\mathbb{Q}_{\geq 0} S_n$.

Thank you!!!