Schur-Positive Symmetric Functions From Polynomial Products

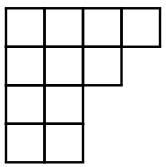
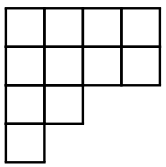
Isaiah Siegl
University of Washington

CAGE Seminar

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Partitions and Tableaux

- A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq 0)$ of n is a list of positive integers that sum to n.

$$\lambda = (4,3,2,2) \longleftrightarrow$$  $$\qquad \lambda' = (4,4,2,1) \longleftrightarrow$$ 

- A (semistandard) Young tableau of shape λ is a filling of λ with positive integers such that columns strictly increase and rows weakly increase.

1	1	2	5
2	3	3	
5	5		
6	7		

- $\mathrm{SSYT}(\lambda) =$ semistandard Young tableaux
- $\mathrm{SYT}(\lambda) =$ Young tableaux with each element of $[n]$ used once.

Symmetric Function Bases

- A powerseries $f(\mathbf{x})$ is *symmetric* if $f(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots)$ for all $i \in \mathbb{Z}_{>0}$.
- Monomial symmetric functions

$$m_\lambda(\mathbf{x}) = \sum_{\substack{\alpha \vDash n \\ \mathrm{sort}(\alpha)=\lambda}} \prod_{i \geq 1} x_i^{\alpha_i}.$$

- Elementary symmetric functions

$$e_k(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad e_\lambda(\mathbf{x}) = e_{\lambda_1}(\mathbf{x}) e_{\lambda_2}(\mathbf{x}) \cdots e_{\lambda_l}(\mathbf{x}).$$

- Schur symmetric functions

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} x_T$$

Real Rooted Polynomials

Theorem (Stanley '98)

Let $F(t) = 1 + a_1 t + \cdots + a_d t^d$ be a polynomial with real coefficients. The following are equivalent.

- $\Omega_F(\mathbf{x}) = \prod_{i \geq 1} F(x_i)$ *is Schur-positive.*
- $\Omega_F(\mathbf{x}) = \prod_{i \geq 1} F(x_i)$ *is e-positive.*
- *All roots of $F(t)$ are negative real numbers.*
- *The matrix $[a_{j-i}]_{i,j=1}^{\infty}$ is totally non-negative.*

Question

What if $F(t)$ has coefficients in $\mathbb{R}[z_1, z_2, \ldots, z_n]$?

Dual Cauchy Product

For $\mathbf{u} = (u_1, u_2, ...)$, the *dual Cauchy product* is the identity

$$\Omega(\mathbf{u}, \mathbf{x}) = \prod_{i,j \geq 1} (1 + u_i x_j) = \prod_{j \geq 1} \sum_{k \geq 0} e_k(\mathbf{u}) x_j^k$$
$$= \sum_{\lambda} e_\lambda(\mathbf{u}) m_\lambda(\mathbf{x}) = \sum_{\lambda} s_\lambda(\mathbf{u}) s_{\lambda'}(\mathbf{x}) = \sum_{\lambda} m_\lambda(\mathbf{u}) e_\lambda(\mathbf{x}).$$

- Let $F(t) = 1 + a_1 t + \cdots + a_d t^d = \prod(1 + \alpha_i t)$. Then $\Omega_F(\mathbf{x}) := \prod_{i \geq 1} F(x_i) = \Omega(\alpha, \mathbf{x})$.
- If $\alpha \in \mathbb{R}^d_{>0}$, then $\Omega_F(\mathbf{x})$ is Schur-positive and e-positive.
- For $\lambda = (\lambda_1, ..., \lambda_\ell)$, $s_{\lambda'}(\mathbf{u}) = \Delta_{I,J}[e_{j-i}(\mathbf{u})]_{i,j=1}^{\infty}$, where $I = \{1, 2, \ldots, \ell\}$ and $J = \{\lambda_\ell, \lambda_{\ell-1} + 1, \ldots, \lambda_1 - \ell + 1\}$.

F-Analogues

- For a polynomial $F(\mathbf{z}, t) \in \mathbb{R}[z_1, z_2, \ldots, z_n][t]$, write $F(\mathbf{z}, t) = 1 + e_1^F(\mathbf{z})t + e_2^F(\mathbf{z})t^2 + \cdots + e_d^F(\mathbf{z})t^d$.
- For a symmetric function $g(\mathbf{x}) = p(e_1, e_2, \ldots)$, define the F-analogue $g^F(\mathbf{z}) = p(e_1^F, e_2^F, \ldots, e_d^F)$.

$$\Omega_F(\mathbf{z}, \mathbf{x}) := \prod_{i \geq 1} F(\mathbf{z}, x_i) = \prod_{i \geq 1} \sum_{k \geq 0} e_k^F(\mathbf{z}) x_i^k$$
$$= \sum_{\lambda} e_\lambda^F(\mathbf{z}) m_\lambda(\mathbf{x}) = \sum_{\lambda} s_\lambda^F(\mathbf{z}) s_{\lambda'}(\mathbf{x}) = \sum_{\lambda} m_\lambda^F(\mathbf{z}) e_\lambda(\mathbf{x}).$$

- If $\lambda = (\lambda_1, ..., \lambda_\ell)$, then $s_{\lambda'}^F(\mathbf{z}) = \Delta_{I,J}[e_{j-i}^F(\mathbf{z})]_{i,j=1}^{\infty}$, where $I = \{1, 2, \ldots, \ell\}$ and $J = \{\lambda_\ell, \lambda_{\ell-1} + 1, \ldots, \lambda_1 + \ell - 1\}$.

Symmetric Function Evaluations

- For $\mathbf{r} \in \mathbb{R}^n$, write $F(\mathbf{r}, t) = \prod_{i=1}^{d}(1 + u_i^{\mathbf{r}} t)$. Then for any symmetric function $g(\mathbf{x})$, we have $g^F(\mathbf{r}) = g(\mathbf{u}^{\mathbf{r}})$.
- Suppose $\Omega_F(\mathbf{z}, \mathbf{x})$ is Schur-positive. (e-positive)
  - If $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$, then $\mathbf{u}^{\mathbf{r}} \in \mathbb{R}^d_{\geq 0}$.
  - $s_\lambda^F(\mathbf{z}) \in \mathbb{R}_{\geq 0}[\mathbf{z}, t]$ for all λ. $\qquad (m_\lambda^F(\mathbf{z}) \in \mathbb{R}_{\geq 0}[\mathbf{z}, t])$.
  - If $\mathbf{r} \geq_{entry} \mathbf{s} \in \mathbb{R}^n_{\geq 0}$, then for all λ,

$$s_\lambda(\mathbf{u}^{\mathbf{r}}) = s_\lambda^F(\mathbf{r}) \geq s_\lambda^F(\mathbf{s}) = s_\lambda(\mathbf{u}^{\mathbf{s}})$$
$$(m_\lambda(\mathbf{u}^{\mathbf{r}}) = m_\lambda^F(\mathbf{r}) \geq m_\lambda^F(\mathbf{s}) = m_\lambda(\mathbf{u}^{\mathbf{s}}).)$$

- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_{\geq 0}$, under what conditions will $s_\lambda(\mathbf{u}) \geq s_\lambda(\mathbf{v})$ for all λ?

Evaluation Inequalities

- For $\mathbf{u} \in \mathbb{R}^d_{\geq 0}$, write $u_{[i]}$ for the i^{th} largest entry of $\mathbf{u}$.
- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_{\geq 0}$, we say $\mathbf{u}$ *log-majorizes* $\mathbf{v}$ and write $\mathbf{u} \succeq_{\log} \mathbf{v}$ if for all $1 \leq k \leq d$,

$$\prod_{i=1}^{k} u_{[i]} \geq \prod_{i=1}^{k} v_{[i]}.$$

Theorem (S.)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_{\geq 0}$. The following are equivalent.

- $\mathbf{u} \succeq_{\log} \mathbf{v}$.
- $s_\lambda(\mathbf{u}) \geq s_\lambda(\mathbf{v})$ *for all partitions* λ.
- $m_\lambda(\mathbf{u}) \geq m_\lambda(\mathbf{v})$ *for all partitions* λ.

The Log-Majorization Property

- For $F(\mathbf{z}, t) \in \mathbb{R}[\mathbf{z}, t]$, write $F(\mathbf{r}, t) = \prod_{i=1}^{d}(1 + u_i^{\mathbf{r}} t)$. Say $F(\mathbf{z}, t)$ satisfies the *log-majorization property* if
  - $\mathbf{u}^{\mathbf{r}} \in \mathbb{R}_{\geq 0}^{d}$ for all $\mathbf{r} \in \mathbb{R}_{\geq 0}^{n}$.
  - If $\mathbf{r} \geq_{entry} \mathbf{s} \in \mathbb{R}_{\geq 0}^{n}$, then $\mathbf{u}^{\mathbf{r}} \succeq_{\log} \mathbf{u}^{\mathbf{s}}$.

Corollary (S.)

If $\Omega_F(\mathbf{z}, \mathbf{x})$ is Schur-positive, then $F(\mathbf{z}, t)$ satisfies the log-majorization property.

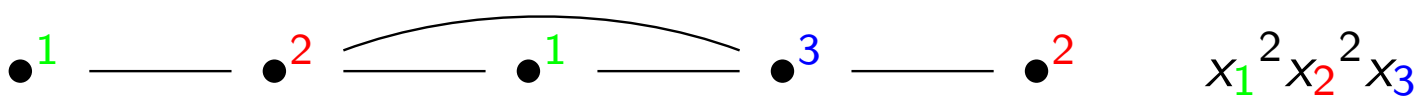
- Log-majorization is weaker than interlacing $(u_{[1]} \geq v_{[1]} \geq u_{[2]} \geq \cdots \geq u_{[d]} \geq v_{[d]})$.

Chromatic Symmetric Functions

Definition (Stanley '95)

For a graph $G = (V, E)$ define the chromatic symmetric function $X_G(\mathbf{x})$ by

$$X_G(\mathbf{x}) = \sum_{\substack{\kappa: V \to \mathbb{Z}_{>0} \\ \kappa(v) \neq \kappa(w) \text{ if } \{v,w\} \in E}} \prod_{v \in V} x_{\kappa(v)}.$$



$$x_1^2 x_2^2 x_3$$

Multivariate Independence Polynomial

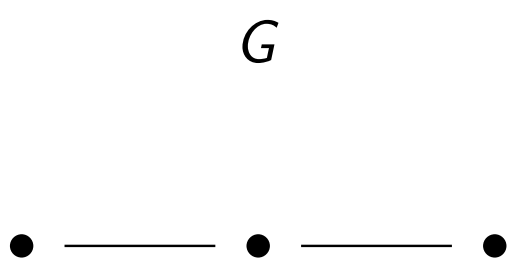
Definition

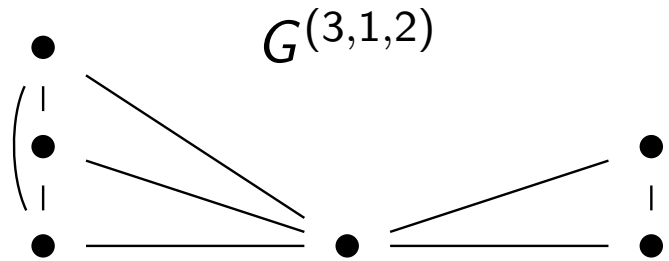
Define the multivariate independence polynomial $I_G(\mathbf{z}, t)$ *of a graph* G *by* $I_G(\mathbf{z}, t) = \sum_S t^{|S|} \prod_{v \in S} z_v$, *where the sum over all independent sets of* G.

Theorem (Stanley '98)

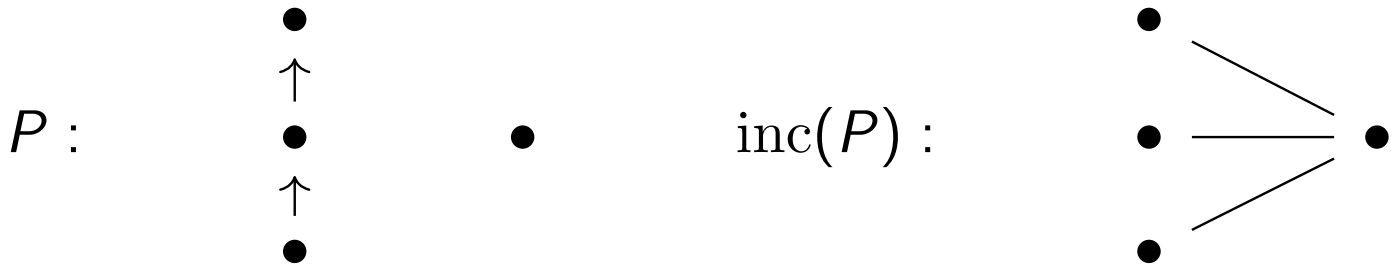
For a graph G *with vertex set* $[n]$ *and* $\alpha \in \mathbb{Z}^n_{\geq 0}$, *write* G^α *for the graph obtained by replacing vertex* i *with a clique of size* α_i. *Then*

$$\prod_{i \geq 1} I_G(\mathbf{z}, x_i) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{\mathbf{z}^\alpha}{\alpha!} X_{G^\alpha}(\mathbf{x}).$$





Incomparability graphs



- For a poset P on $[n]$, the *incomparability graph* $\mathrm{inc}(P)$ is the graph on $[n]$ with edges $\{i, j\}$ for i, j incomparable in P.
- A poset P is $(\mathbf{3}+\mathbf{1})$-free if it has no chain of three elements which are all incomparable to some fourth element.
- Equivalently, $\mathrm{inc}(P)$ is claw-free.

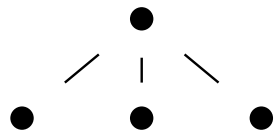
Schur Positivity

- Let P be a $(\mathbf{3}+\mathbf{1})$-free poset
- P-tableaux $\mathrm{T}_P(\lambda)$: filling T of λ with elements of P such that
  - $T_{i,j} \not>_P T_{i,j+1}$ (non P-descending rows)
  - $T_{i,j} <_P T_{i+1,j}$ (P-increasing columns)
- $\mathrm{ST}_P(\lambda)$: each element of P appears exactly once.
- **Theorem**[Gasharov '96]

$$X_{\mathrm{inc}(P)}(\mathbf{x}) = \sum_{\lambda \vdash n} |\mathrm{ST}_P(\lambda)| s_{\lambda'}(\mathbf{x}).$$

- **Corollary**[Stanley '98]: All roots of the independence polynomial $I_{\mathrm{inc}(P)}(\mathbf{r}, t)$ are negative real numbers for all $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$.
- **Corollary**[S '26]: $I_{\mathrm{inc}(P)}(\mathbf{z}, t)$ satisfies the log-majorization property.

Claw-Free Graphs

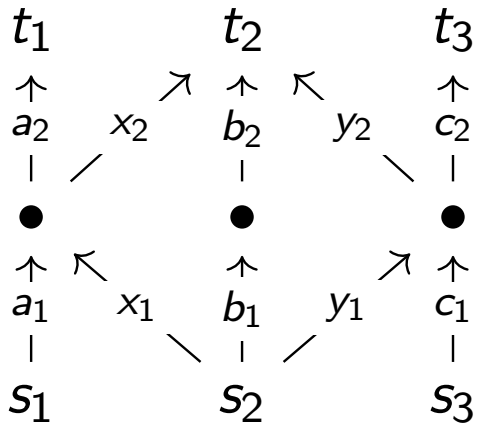
- G is *claw-free* if it does not have an induced claw 
- **Theorem**[Chudnovsky–Seymour '07, Leake–Ryder '19] Let G be a claw-free graph on $[n]$. For all $\mathbf{r} \in \mathbb{R}^n_{\geq 0}$, all roots of $I_G(\mathbf{r}, t)$ are negative real numbers.
- **Conjecture**[Stanley–Gasharov '98] If G is claw-free, then $X_G(\mathbf{x})$ is Schur-positive.
- **Counterexample**[Matherne–Morales '26, Prajapati '26]: There is a claw-free graph on 12 vertices with non-Schur-positive $X_G(\mathbf{x})$.
- Does $I_G(\mathbf{z}, t)$ have the log-majorization property for all claw-free graphs G?

Hessenberg Functions

- A *Hessenberg function* is map $\mathbf{h} : [n] \to [n]$ such that
  - $h_i \leq h_{i+1}$ for all $1 \leq i < n$, and
  - $h_i \geq i$ for all $1 \leq i \leq n$.
- Define $P_{\mathbf{h}}$ to be the poset such that $i <_{P_{\mathbf{h}}} j$ if $h_i < j$.
- $P_{\mathbf{h}}$ is $(\mathbf{3}+\mathbf{1})$-free
- **Theorem**[Guay-Paquet '13] If P is $(\mathbf{3}+\mathbf{1})$-free then $X_{\mathrm{inc}(P)}(\mathbf{x})$ is a convex combination of $\{X_{\mathrm{inc}(P_{\mathbf{h}})}(\mathbf{x}) | \, \mathbf{h} \text{ Hessenberg}\}$
- Write $I_{\mathrm{inc}(P_{\mathbf{h}})}(\mathbf{z}, t) = \sum_{k \geq 0} e_k^{\mathbf{h}}(\mathbf{z}) t^k$.
- If $\left[e_{j-i}^{\mathbf{h}}(\mathbf{z}) \right]_{i,j=1}^{\infty}$ is totally non-negative, then $X_{\mathrm{inc}(P_{\mathbf{h}})}(\mathbf{x})$ is Schur-positive.

Lindström–Gessel–Viennot

- Directed planar networks correspond to totally nonnegative matrices
- Minors of TNN matrices correspond to non-intersecting path families



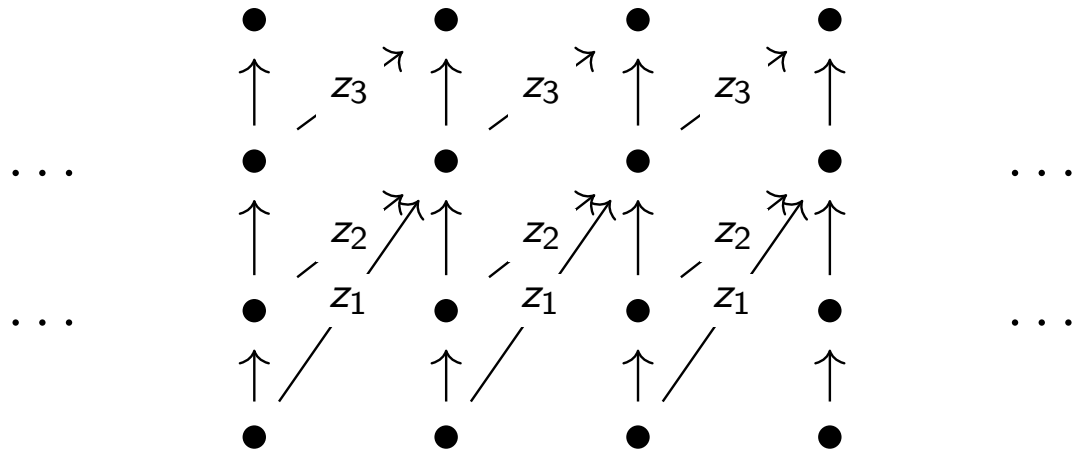
$$\leftrightarrow \begin{bmatrix} a_1a_2 & a_1x_2 & 0 \\ x_1a_2 & x_1x_2 + b_1b_2 + y_1y_2 & y_1c_2 \\ 0 & c_1y_2 & c_1c_2 \end{bmatrix}$$

- Can we find a directed planar network that proves total non-negativity of $\left[e^{\mathbf{h}}_{j-i}(\mathbf{z})\right]_{i,j=1}^{\infty}$?

$\Gamma_{\mathbf{h}}$

- Let $\mathbf{h} : [n] \to [n]$ be a Hessenberg function.
- Define $\Gamma_{\mathbf{h}}$ be the directed graph on $\mathbb{Z} \times [n+1]$ with
  - Edges of weight 1: $(i,j) \to (i,j+1)$
  - Edges of weight z_j: $(i,j) \to (i+1, h_j+1)$.

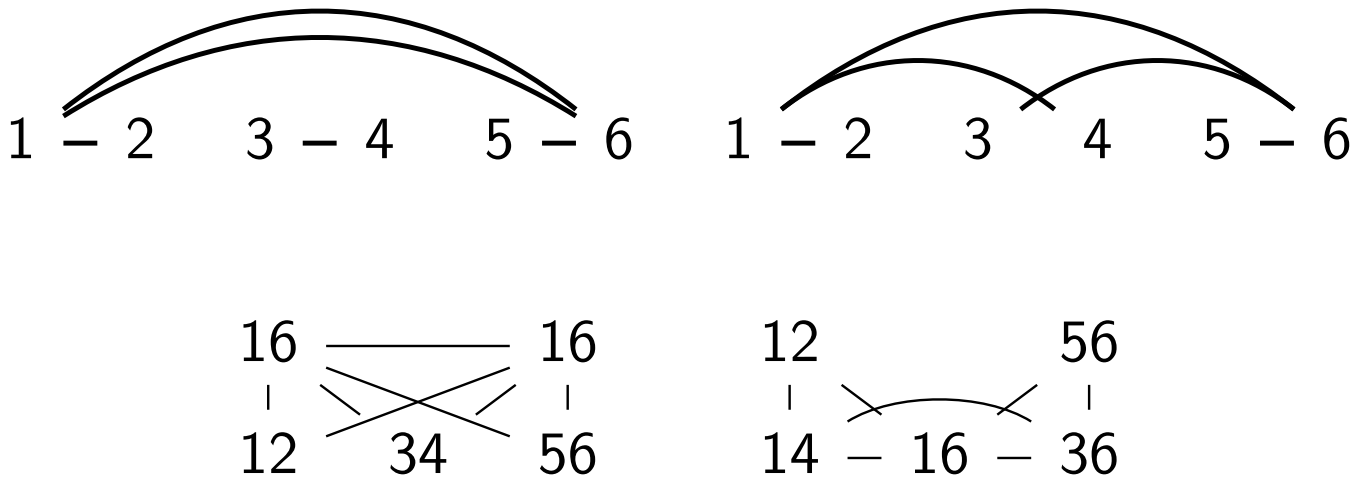
$\Gamma_{(2,2,3)}$:



- The weight matrix of $\Gamma_{\mathbf{h}}$ is $[e^{\mathbf{h}}_{j-i}(\mathbf{z})]_{i,j=-\infty}^{\infty}$.
- New proof of Schur-positivity of $X_{\mathrm{inc}(P_{\mathbf{h}})}(\mathbf{x})$.
- Do networks exist for other $(\mathbf{3}+\mathbf{1})$-free posets?

Sums of Chromatic Symmetric Functions

- Let $M = \{(i <_{P_{\mathbf{h}}} j)\}$ be a multiset of pairs of elements of $P_{\mathbf{h}}$.
- Let $\deg(M)_i^-$ ($\deg(M)_i^+$) be the number of times i appears as the smaller (larger) element in M.
- Let G_M be the graph on M with edges between overlapping pairs.
- Ex. $\deg(M)^- = (3, 0, 1, 0, 1, 0)$, $\deg(M)^+ = (0, 1, 0, 1, 0, 3)$, $\mathbf{h} = (1, 2, 3, 4, 5, 6)$.



$Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x})$

- For a multiset M, write $m_e(M)$ for the multiplicity of an element e.
- For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n_{\geq 0}$, let

$$Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x}) = \sum_{\substack{\deg(M)^- = \mathbf{a} \\ \deg(M)^+ = \mathbf{b}}} \prod_{i <_P j} \frac{1}{m_{\{i,j\}}(M)!} X_{G_M}(\mathbf{x}).$$

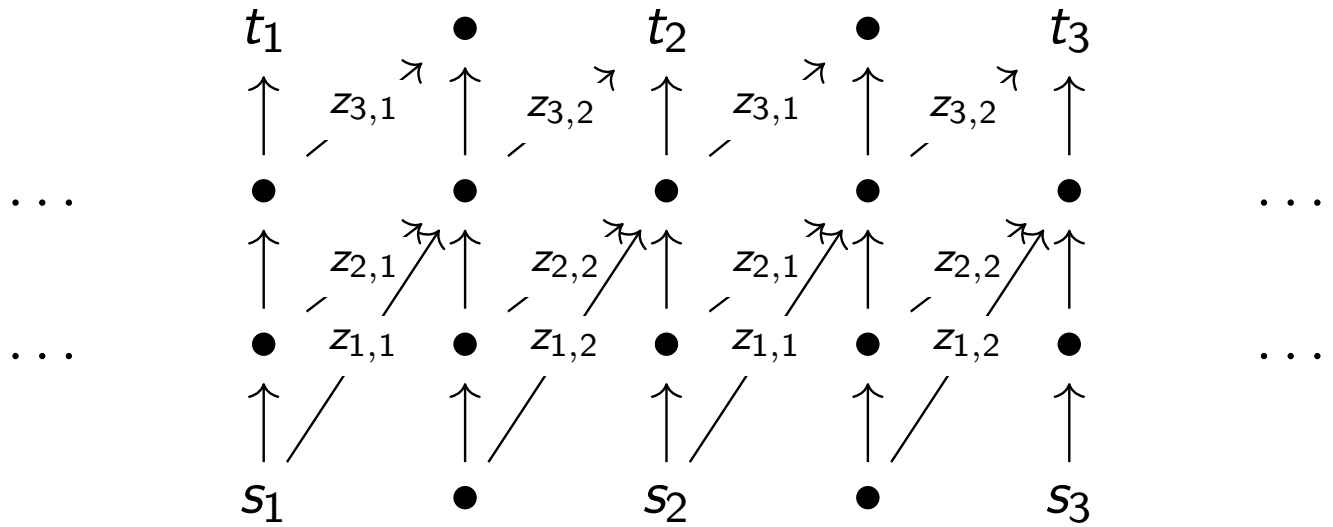
- Ex. $\mathbf{a} = (3,0,1,0,1,0)$, $\mathbf{b} = (0,1,0,1,0,3)$, $\mathbf{h} = (1,2,3,4,5,6)$. $M_1 = \{16,16,12,34,56\}$ and $M_2 = \{12,14,16,56,36\}$

$$\begin{aligned} Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x}) &= \frac{1}{2} X_{G_{M_1}}(\mathbf{x}) + X_{G_{M_2}}(\mathbf{x}) \\ &= \frac{1}{2}(2s_{3,1,1}(\mathbf{x}) - 2s_{2,2,1}(\mathbf{x}) + 16s_{2,1,1,1}(\mathbf{x}) + 54s_{1,1,1,1,1}(\mathbf{x})) \\ &+ (2s_{2,2,1}(\mathbf{x}) + 14s_{2,1,1,1}(\mathbf{x}) + 54s_{1,1,1,1,1}(\mathbf{x})). \end{aligned}$$

$Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x})$ is Schur-positive

- **Theorem**[S.] For all $\mathbf{a}, \mathbf{b}, \mathbf{h}$, we have that $Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x})$ is Schur-positive.
- Let $e_k^{\mathbf{h},(2)}(\mathbf{z}) = \sum_{i_1 <_{P_\mathbf{h}} i_2 <_{P_\mathbf{h}} \ldots <_{P_\mathbf{h}} i_{2k}} \prod_{j=1}^{k} z_{i_{2j-1},1} z_{i_{2j},2}$, then

$$\prod_{i\geq 1} \sum_{k\geq 0} e_k^{\mathbf{h},(2)}(\mathbf{z}) x_i^k = \sum_{\mathbf{a},\mathbf{b}\in\mathbb{Z}_{\geq 0}^n} Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x}) \prod_{i=1}^{n} z_{i,1}^{a_i} z_{i,2}^{b_i}.$$



e-positivity

- **Theorem**[Hikita '24, Conj. Stanley–Stembridge '93] For any $(\mathbf{3}+\mathbf{1})$-free poset P, $X_{\mathrm{inc}(P)}(\mathbf{x})$ is e-positive.
- Let $\mathcal{M}_n$ be the set of perfect matchings on $[2n]$. Define

$$PM_n(\mathbf{x}) = \sum_{M \in \mathcal{M}_n} X_{G_M}(\mathbf{x}).$$

- **Theorem**[Ambdeberhan–Shareshian–Stanley '26] For all n, $PM_n(\mathbf{x})$ is e-positive.

$$\prod_{i \geq 1} \sec(\sqrt{x_i}) = \omega \sum_{n \geq 0} \frac{1}{(2n)!} PM_n(\mathbf{x}).$$

e-Positivity of $Y_{\mathbf{a},\mathbf{b},\mathbf{h}}$

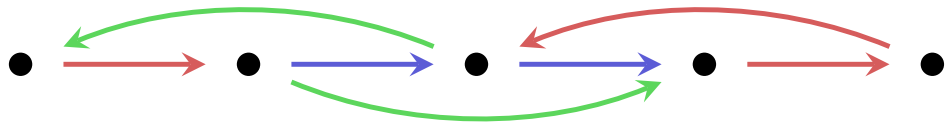
- Let $\mathbf{h} = (1, 2, ..., 2n)$. Then

$$PM_n(\mathbf{x}) = \sum_{\substack{\mathbf{a},\mathbf{b}\in\{0,1\}^{2n} \\ \mathbf{a}+\mathbf{b}=\mathbf{1}_{[2n]}}} Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x}).$$

- **Conjecture**[S.] $Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x})$ is e-positive for all $\mathbf{a}, \mathbf{b}, \mathbf{h}$.
- Ex. $\mathbf{a} = (3, 0, 1, 0, 1, 0)$, $\mathbf{b} = (0, 1, 0, 1, 0, 3)$, $\mathbf{h} = (1, 2, 3, 4, 5, 6)$. $M_1 = \{16, 16, 12, 34, 56\}$ and $M_2 = \{12, 14, 16, 56, 36\}$

$$\begin{aligned} Y_{\mathbf{a},\mathbf{b},\mathbf{h}}(\mathbf{x}) &= \frac{1}{2}X_{G_{M_1}}(\mathbf{x}) + X_{G_{M_2}}(\mathbf{x}) \\ &= \frac{1}{2}(2e_{3,1,1}(\mathbf{x}) - 4e_{3,2}(\mathbf{x}) + 16e_{4,1}(\mathbf{x}) + 40e_5(\mathbf{x})) \\ &+ (2e_{3,2}(\mathbf{x}) + 12e_{4,1}(\mathbf{x}) + 40e_5(\mathbf{x})). \end{aligned}$$

Kirchhoff Symmetric Functions



- For a graph $G = (V, E)$, a *rooted forest* is a pair (A, R) such that
  - $A \subseteq E$ is a forest, and
  - $R \subseteq V$ has exactly one vertex from each connected component of A.
- A rooted forest partition of G of shape λ is a sequence of rooted forests $(A_i, R_i)_{i=1}^{\ell}$ such that $|A_i| = \lambda_i$ and $\bigcup_{i=1}^{\ell} A_i = E$.
- Write $\Pi_G(\lambda)$ for the set of rooted forest partitions of G of shape λ.
- Define the *Kirchhoff symmetric function* $K_G(\mathbf{x})$ of G by

$$K_G(\mathbf{x}) = \sum_{\lambda \vdash |E|} |\Pi_G(\lambda)| m_\lambda(\mathbf{x}).$$

$K_G(\mathbf{x})$ From Products

- Let $\mathcal{RF}(G)$ be the set of rooted forests of G, and let

$$RF_G(\mathbf{z}, t) = \sum_{(A,R)\in\mathcal{RF}(G)} t^{|A|} \prod_{e\in A} z_e.$$

- For $\alpha \in \mathbb{Z}^E_{\geq 0}$, let $G[\alpha]$ be the graph with α_e copies of edge e.
- **Lemma**[S.] For a loopless graph G,

$$\prod_{i\geq 1} RF_G(\mathbf{z}, x_i) = \sum_{\alpha\in\mathbb{Z}^E_{\geq 0}} \frac{\mathbf{z}^\alpha}{\alpha!} K_{G[\alpha]}(\mathbf{x}).$$

- **Theorem**[S.] For a loopless graph G, $K_G(\mathbf{x})$ is Schur-positive.

The Matrix-Tree Theorem

- For $e = \{i, j\} \in E(G)$, let $v_e = \mathbf{1}_i - \mathbf{1}_j \in \mathbb{R}^V$.
- Define the *weighted Laplacian* $\mathcal{L}_G(\mathbf{z}) = \sum_{e \in E} z_e v_e v_e^* \in \mathbb{R}[\mathbf{z}]^{V \times V}$.
- **Theorem**(Matrix-Tree) For a graph G,

$$RF_G(\mathbf{z}, t) := \sum_{(A,R) \in \mathcal{RF}(G)} t^{|A|} \prod_{e \in A} z_e = \det(I + t\mathcal{L}_G(\mathbf{z})).$$

- For a square matrix A and $g(\mathbf{x}) \in \Lambda$, write $g[A]$ for the $g(\mathbf{x})$ evaluated at the eigenvalues of A.
- $\prod_{i \geq 1} \det(I + x_i A) = \sum_\lambda e_\lambda[A] m_\lambda(\mathbf{x}) = s_\lambda[A] s_{\lambda'}(\mathbf{x})$.

$GL(V)$ Representations

- For a finite dimensional vector space V and partition $\lambda \vdash m$, write $S_\lambda(V)$ for the irreducible $GL(V)$-module indexed by λ.
- $V^{\otimes m} \cong \bigoplus_{\lambda \vdash m} S_\lambda(V)^{\oplus f^\lambda}$.
- Let $P_\lambda : V^{\otimes m} \to V^{\otimes m}$ be the orthogonal projection onto $S_\lambda(V)^{\oplus f^\lambda}$.
- For $A : V \to V$, $s_\lambda[A] = \mathrm{Tr}_{S_\lambda(V)}(A) = \frac{1}{f^\lambda}\mathrm{Tr}(P_\lambda A^{\otimes m})$.
- For $A(\mathbf{z}) = \sum_{i=1}^n z_i v_i v_i^*$ for some $v_i \in V$, then

$$s_\lambda[A(\mathbf{z})] = \frac{1}{f^\lambda}\mathrm{Tr}(P_\lambda A^{\otimes m}) = \frac{1}{f^\lambda}\mathrm{Tr}(P_\lambda \sum_{\mathbf{w}\in[n]^m} z_{\mathbf{w}} v_{\mathbf{w}} v_{\mathbf{w}}^*)$$

$$= \frac{1}{f^\lambda}\sum_{\mathbf{w}\in[n]^m} \mathrm{Tr}(P_\lambda z_{\mathbf{w}} v_{\mathbf{w}} v_{\mathbf{w}}^*) = \frac{1}{f^\lambda}\sum_{\mathbf{w}\in[n]^m} z_{\mathbf{w}}(P_\lambda v_{\mathbf{w}})^*(P_\lambda v_{\mathbf{w}}) \in \mathbb{R}_{\geq 0}[\mathbf{z}],$$

where $z_{\mathbf{w}} = \prod_{i=1}^m z_{w_i}$ and $v_{\mathbf{w}} = v_{w_1} \otimes v_{w_2} \otimes \cdots \otimes v_{w_m}$.

Positivity for PSD matrices

- If $A(\mathbf{z}) = \sum_{i=1}^{n} z_i v_i v_i^* \in \mathbb{C}[\mathbf{z}]^{d\times d}$, then

$$s_\lambda[A(\mathbf{z})] = \frac{1}{f^\lambda} \sum_{\mathbf{w}\in[n]^m} z_{\mathbf{w}} (P_\lambda v_{\mathbf{w}})^* (P_\lambda v_{\mathbf{w}}) \in \mathbb{R}_{\geq 0}[\mathbf{z}].$$

- $\prod_{i\geq 1} \det(I + x_i A(\mathbf{z})) = \sum_\lambda s_\lambda[A(\mathbf{z})] s_{\lambda'}(\mathbf{x})$.
- Define the λ-immanant $\mathrm{Imm}_\lambda(A) = \sum_{\pi\in S_d} \chi^\lambda(\pi) \prod_{i=1}^{d} A_{i,\pi(i)}$.
- For $\mathbf{w} \in [n]^m$, $(P_\lambda v_{\mathbf{w}})^* (P_\lambda v_{\mathbf{w}}) = \mathrm{Imm}_\lambda([v_{w_i}^* v_{w_j}]_{i,j=1}^m)$.
- Do the Schur-coefficients of $K_G(\mathbf{x})$ have a nice combinatorial interpretation?
- Are there other combinatorially interesting matrices of the form $A(\mathbf{z}) = \sum_{i=1}^{n} z_i v_i v_i^*$?

Other Coefficient Rings

- Let $\mathcal{R}$ be a not-necessarily-commutative $\mathbb{Q}$-algebra, and let $\mathcal{R}_+ \subset \mathcal{R}$ be closed under multiplication and $\mathbb{Q}_{\geq 0}$-linear combinations.
- For $F(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_d t^d \in \mathcal{R}$, the coefficients $\{a_i\}$ must commute pairwise for $\prod_{i \geq 1} F(x_i)$ to be symmetric.
- Ex. [Fomin–Greene '98] Let $\mathcal{R} = \mathbb{Q}\langle z_1, z_2, \ldots, z_n \rangle / I$ where $I = (z_b z_a z_c - z_b z_c z_a | a < b \leq c) + (z_c z_a z_b - z_a z_c z_b | a \leq b < c)$. Then the Schur-coefficients of

$$\prod_{i \geq 1} \sum_{k \geq 0} x_i^k \sum_{i_1 > \cdots > i_k \geq 1} z_{i_1} z_{i_2} \cdots z_{i_k}$$

have positive representatives in $\mathbb{Q}\langle z_1, z_2, \ldots z_n \rangle / I$.

$\mathcal{R} = \mathbb{Q}S_n$

- For $\pi \in S_n$, $\mathrm{des}(\pi) = |\{i \in [n-1] : \pi(i) > \pi(i+1)\}|$.
- Define the n^{th} Eulerian polynomial $A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}$.
- $A_n(t)$ is real-rooted.
- **Theorem**[Loday '89] Let $A_n^{(i)} = \sum_{\mathrm{des}(\pi)=i} \pi \in \mathbb{Q}S_n$. Then $A_n^{(i)} A_n^{(j)} = A_n^{(j)} A_n^{(i)}$ for all $0 \leq i, j < n$.
- **Conjecture**[S.] Let $\mathcal{A}_n(t) = \sum_{i=1}^{n-1} A_n^{(i)} t^i = \sum_{\pi \in S_n} \pi \, t^{\mathrm{des}(\pi)}$. Then the Schur-coefficients of $\prod_{i \geq 1} \mathcal{A}_n(x_i)$ are in $\mathbb{Q}_{\geq 0} S_n$.

Thank you!!!