Equidistribution of Weierstrass points on tropical curves

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Weierstrass points

The **Weierstrass locus** $W(D)$ of a divisor D on an algebraic curve X consists of points of "higher than expected tangency" with hyperplanes in the projective embedding $\phi_D : X \to \mathbb{P}^r$,

$$W(D) = \{x \in X : \phi(X) \cap H \geq (r+1)x \text{ for some hyperplane } H\}.$$

On a genus 1 curve, these are the N-torsion points (up to some translation).

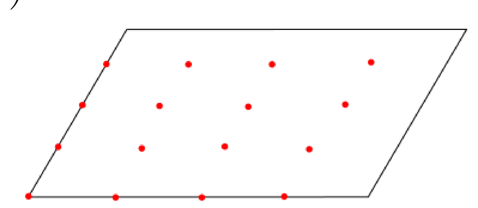

Figure 1: 4-torsion points on a complex elliptic curve

As $N \to \infty$, N-torsion points "evenly distribute" over a complex elliptic curve. In general, Mumford suggested we should consider Weierstrass points as higher-genus analogues of N-torsion points. This makes it natural to ask:

Problem

How do Weierstrass points distribute on a curve?

For curves over $\mathbb{C}$, this was answered by Amnon Neeman, a student of Mumford.

Theorem (Neeman, 1984) If X is complex algebraic curve, the Weierstrass points $W(D_N)$ distribute according to the Bergman measure on X as $N \to \infty$.

We can also consider curves over a non-Archimedean field $(\mathbb{K}, \mathrm{val} : \mathbb{K}^\times \to \mathbb{R})$, which we assume is algebraically closed. The Weierstrass points lie in $X(\mathbb{K}) \subset X^{\mathrm{an}}$.

Theorem (Amini, 2014) If X^{an} is Berkovich curve, the Weierstrass points $W(D_N)$ distribute according to the Zhang measure on X^{an} as $N \to \infty$.

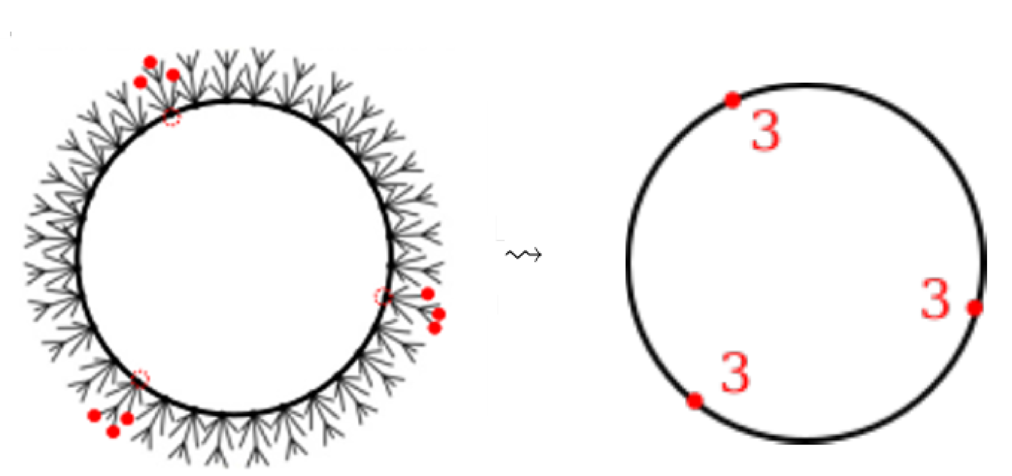

Figure 2: Weierstrass points on Berkovich elliptic curve and its skeleton

Tropical curves

A **tropical curve** is a metric space obtained from a finite graph by assigning edge lengths. Geometrically, it can represent a smooth algebraic curve degenerating to a collection of $\mathbb{P}^1$'s meeting at nodes. We turn a degeneration $X_t \rightsquigarrow X_0$ into a metric graph by making each $\mathbb{P}^1$-component of X_0 into a vertex and each node of X_0 into an edge, whose length is equal to the "rate of degeneration" of the node. Explicitly, we assign length L to the node $\{uv - t^L = 0\}$.

Example: $X_t = \{xyz + tx^3 + t^2y^3 + t^5z^3 = 0\} \subset \mathbb{P}^2(\mathbb{C})$

The node $\{x, z = 0\}$ is assigned an edge of length 2 in the dual graph, since the node is described by $\{uv + t^2\}$ in a local-analytic neighborhood. The nodes $\{x, y = 0\}$ and $\{y, z = 0\}$ are assigned edge lengths 5 and 1 respectively.

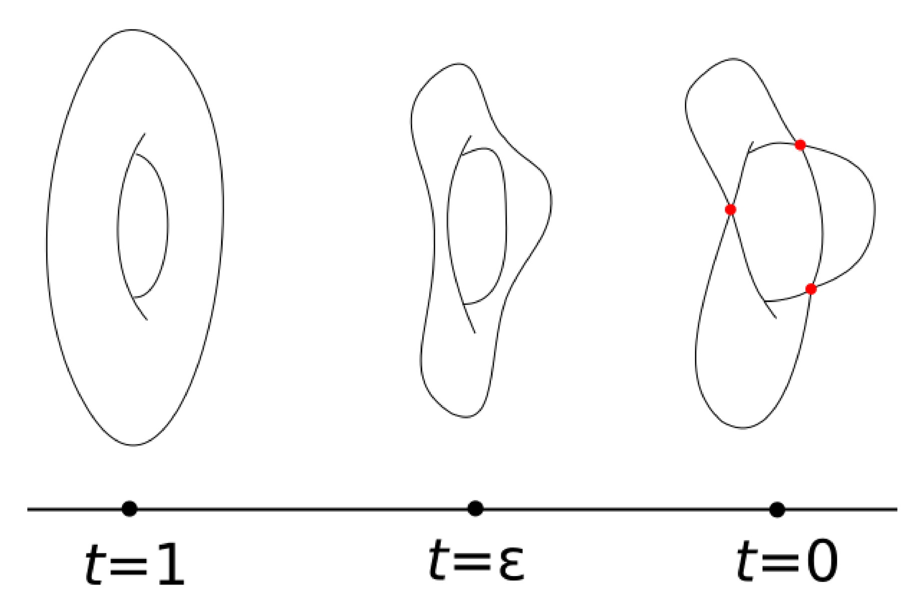

Figure 3: Elliptic curve degenerating to nodal curve with three $\mathbb{P}^1$ components

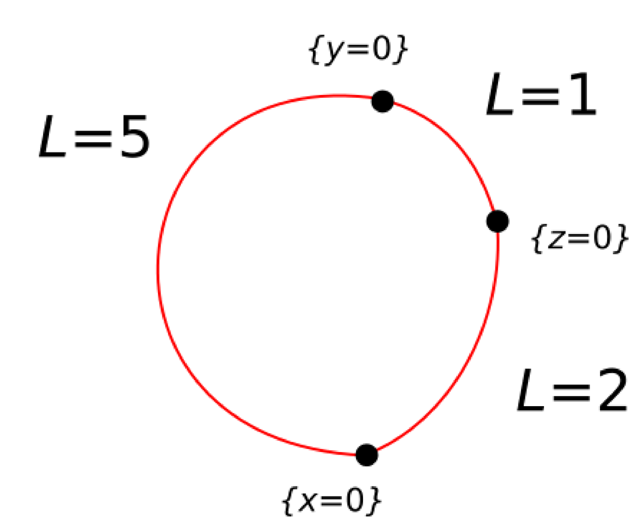

Figure 4: Dual metric graph of degeneration

A one-parameter family X_t of curves over $\mathbb{C}$ is also a single curve over the field of rational functions $\mathbb{C}(t)$, which has non-Archimedean valuation

$$\mathrm{val}(a_0 t^n + a_1 t^{n+1} + \cdots) = n.$$

Chosing different $\mathbb{C}[t]$-models for a curves gives different vertex sets in the resulting dual metric graph.

On a metric graph an (effective) **divisor** D is a finite collection of "chips" placed on Γ. **Linear equivalence** means we may move any subset of chips along a cut-set of Γ, at the same speed and direction. Intuitively, this amounts to "discrete current flow" on Γ.

algebraic curve X		tropical curve Γ
divisors $\mathrm{Div}(X)$	$\rightsquigarrow$	divisors $\mathrm{Div}(\Gamma)$
meromorphic functions	$\rightsquigarrow$	piecewise $\mathbb{Z}$-linear functions
linear system $\lvert D\rvert$ $= \mathbb{P}^r$	$\rightsquigarrow$	linear system $\lvert D\rvert$ $=$ polyhedral complex of dim $\geq r$
rank $r = \dim \lvert D\rvert$	$\rightsquigarrow$	rank $r =$ Baker-Norine rank

Table 1: Divisor theory from algebraic curves to tropical curves

Tropical Weierstrass points

The **tropical Weierstrass locus** $W(D)$ of a divisor on a metric graph Γ is defined as

$$W(D) = \{x \in \Gamma : E \geq (r+1)x \text{ for some } E \in |D|\}$$

where $r = r(D)$ is the Baker-Norine rank. When $\deg(D) \geq 2g - 1$, $r(D) = N - g$.

In Amini's theorem, the limiting distribution μ depends only on a skeleton Γ of X^{an}. Thus it is natural to ask whether this result can be stated for Γ and proved by purely combinatorial methods. However, $W(D)$ is **not always finite** on Γ.

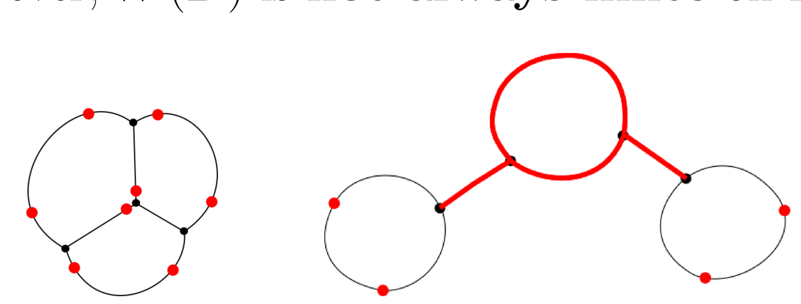

Figure 5: Weierstrass locus $W(K)$ on two genus 3 curves

Reduced divisors

A **reduced divisor** $\mathrm{red}_q[D]$ is the unique representative linearly equivalent to D whose chips are "as close as possible" to $q \in \Gamma$.

Example:

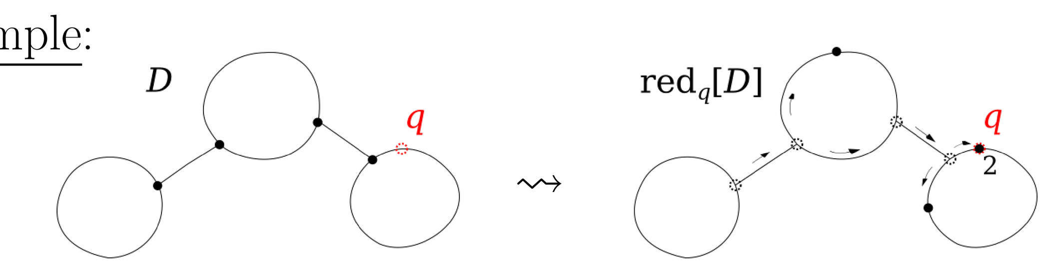

Figure 6: Reduced divisor $\mathrm{red}_q[D]$

Dhar's burning algorithm is an easy method for computing reduced divisors. This allows us to find the Weierstrass locus since

$$x \in W(D) \quad \Leftrightarrow \quad \mathrm{red}_x[D] \geq (r+1)x.$$

Canonical measure

Zhang's canonical measure μ on Γ may be defined in terms of resistor networks, following Baker–Faber. We consider Γ a **resistor network** making each edge a resistor with resistance = length. Given points $y, z \in \Gamma$, we let

$$j_z^y = \left(\begin{array}{l}\text{voltage on } \Gamma \text{ when 1 unit of} \\ \text{current is sent from } y \text{ to } z\end{array}\right)$$

and Γ is "grounded" at z. The **current** through an edge is the slope $|j'|$ of the voltage function (Ohm's law).

Example:

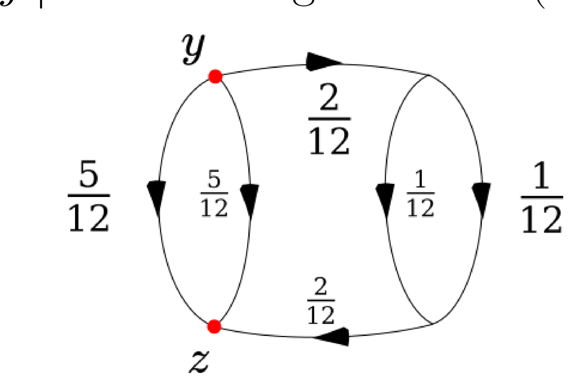

Figure 7: Current flow from y to z on Γ with unit edge lengths

The **canonical measure** $\mu(e)$ is the "current defect"

$$\mu(e) = \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+$$
$$= 1 - (\text{ current through } e \text{ when } \cdots).$$

Example:

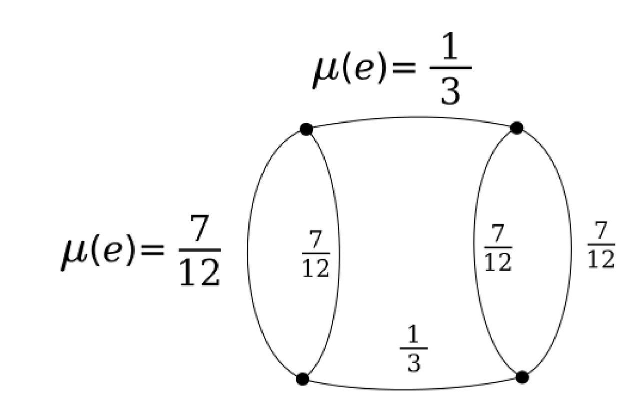

Figure 8: Canonical measures on Γ with unit edge lengths

Results

Theorem A. For a generic divisor class $[D]$ on Γ, the Weierstrass locus $W(D)$ is finite.

Theorem B. Let e be an edge of Γ and let $[D_N]$ be a generic divisor class of degree N. As $N \to \infty$,

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e).$$

where μ is Zhang's canonical measure.

Proof Idea:

$$\begin{array}{ccc}\text{(discrete current flow)} & \xrightarrow{N\to\infty} & \text{(continuous current flow)} \\ \updownarrow & & \updownarrow \\ \#(W(D_N) \cap e) & & \text{canonical measure } \mu(e)\end{array}$$