A dyadic endomorphism which is Bernoulli but not standard

Christopher Hoffman
Daniel Rudolph

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Abstract

Any measure preserving endomorphism generates both a decreasing sequence of σ-algebras and an invertible extension. In this paper we exhibit a dyadic measure preserving endomorphism (X, T, μ) such that the decreasing sequence of σ-algebras that it generates is not isomorphic to the standard decreasing sequence of σ-algebras. However the invertible extension is isomorphic to the Bernoulli two shift.

1 Introduction

Consider the one sided Bernoulli two shift. This transformation has state space $X = \{0, 1\}^{\mathbb{N}}$ and (1/2,1/2) product measure μ. The action on X is $T(x)_i = x_{i+1}$. In this paper we consider two properties that the one sided Bernoulli two shift has and give an example of an endomorphism which shares one of these properties but not the other.

The first property is the decreasing sequence of σ-algebras that the one sided Bernoulli two shift generates. A decreasing sequence of σ-algebras is a measure space $(X, \mathcal{F}_0, \mu)$, and a sequence of σ-algebras $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 ....$ Let $\mathcal{F}$ be the Borel σ-algebra of X and let $\mathcal{F}_i = T^{-i}\mathcal{F}$. This sequence has the property that $\mathcal{F}_i|\mathcal{F}_{i+1}$ has 2 point fibers of equal mass for every i. A decreasing sequence of σ-algebras with this property (and an endomorphism which generates such a sequence) is called **dyadic**. This example has the property that $\cap\mathcal{F}_i$ is trivial. A. Vershik, who began the modern study of such decreasing sequences of σ-algebras [9], refers to this example as the "standard dyadic" example. Any measure preserving endomorphism $(X, T, \mathcal{F}, \mu)$ generates a decreasing sequence of σ-algebras by setting $\mathcal{F}_i = T^{-i}\mathcal{F}$.

Two decreasing sequences of σ-algebras are called **isomorphic** if there exists a 1-1 measure preserving map between the two spaces that carries the i-th σ-algebras to each other. In [8] Vershik showed that there exist dyadic sequences of σ-algebras with trivial intersection that are not isomorphic to the standard dyadic example. In [8] Vershik also gave a necessary and sufficient condition for a dyadic decreasing sequence of σ-algebras to be standard. An equivalent description of standardness for dyadic sequences is for there to exist a sequence of partitions $\{P_i\}$ of X into two sets, each of measure 1/2, such that

1. the partitions P_i are mutually independent and
2. for each i, $\mathcal{F}_i = \bigvee_{n=i}^{\infty} P_n$.

It is important to note that in the case that the decreasing sequence of σ-algebras comes from an endomorphism there is no assumption that the P_i are stationary. (i.e. P_i is not necessarily $T^{-1}(P_{i-1})$.) If an endomorphism generates a decreasing sequence of σ-algebras that is isomorphic to the standard dyadic decreasing sequence of σ-algebras then we call that endomorphism **standard**.

Any measure preserving endomorphism $(X, T, \mathcal{F}, \mu)$ also generates an invertible measure preserving automorphism $(\bar{X}, T, \bar{\mathcal{F}}, \bar{\mu})$. We say that the system $(\bar{X}, T, \bar{\mu})$ is isomorphic to the (invertible) Bernoulli 2 shift if there exists a partition P of $\bar{X}$ into two sets, each of measure 1/2, such that

1. the partitions $T^i P$ are mutually independent and
2. $\bar{\mathcal{F}} = \bigvee_{n=-\infty}^{\infty} T^i P$.

If a dyadic endomorphism has an invertible extension which is isomorphic to the (invertible) Bernoulli 2 shift then we say the endomorphsim is **Bernoulli**.

Both Bernoulliness and standardness are then equivalent to finding a mutually independent sequence of partitions which generate the entire σ-algebra. There is no a priori reason that a standard endomorphism must be Bernoulli or that a Bernoulli endomorphism must be standard as in the case of standardness the partitions must be past measurable but not necessarilly stationary and for Bernoulliness they must be stationary but not necessarilly past measurable. In this paper we show that in fact neither condition is stronger than the other.

It is already known that standard does not imply Bernoulliness of an endomorphism. Feldman and Rudolph proved in [3] that a certain class of dyadic endomorphisms generate standard decreasing sequences of σ-algebras. Among these is an endomorphism which Burton showed has a two sided extension which is not isomorphic to a Bernoulli shift [1]. In this paper we construct a dyadic endomorphsim with entropy log 2 that is Bernoulli but the endomorphism is not standard. In [8] Vershik created an infinite entropy endomorphism which is Bernoulli but not standard. To complete the picture we mention that the one sided Bernoulli 2 shift is both Bernoulli and standard. The $[T, T^{-1}]$ endomorphism was proved not to be Bernoulli by Kalikow [6] and not to be standard by Heicklen and Hoffman [5].

2 Notation

We begin to introduce some notation to help understand the tree structure of a dyadic endomorphism. Consider a rooted 2-ary tree with 2^n vertices at depth $n \geq 0$. Each vertex at depth n connects to two vertices at the depth $n+1$. For each pair of vertices at depth $n+1$ which connect to the same vertex at depth n we label one of the vertices 0 and the other vertex 1. This then gives each vertex a second label which is a nontrivial finite word of zeros and ones. This is given by sequence of values we see along the unique path of vertices from the root to the given vertex. In this form we can concatenate vertices v' and v by concatenating their labels. Call this labeled tree $\mathcal{T}$. If we truncate the tree at depth $n > 0$ we call it $\mathcal{T}_n$.

We also use the notation $v \in \mathcal{T}$ (or $v \in \mathcal{T}_n$) to indicate that v is a vertex of $\mathcal{T}$ (or $\mathcal{T}_n$). set of vertices For $v \in \mathcal{T}$ and at depth i (i.e. $v \in \mathcal{T}_i \setminus \mathcal{T}_{i-1}$) we write $|v| = i$ and we write v as a list of values $v_1, \ldots, v_i$ from $\{0, 1\}$ where this is the list of labels of the vertices along the branch from the root to v. We say that v' is an **extension** of v if $v' = vv''$ for some $v'' \in \mathcal{T}$. We also say that v is a **contraction** of v'.

Let $(X, T, \mu, \mathcal{F})$ be a uniformly 2 to 1 endomorphism. Then each $x \in X$ has two inverse images. There exists a measurable two set partition K of X such that almost every x has one preimage in each element of K. Label the sets of K as K_0 and K_1. For each $i \in \{0, 1\}$ and $x \in X$ define $T_i(x)$ to be the preimage of x in K_i. We now define a set of **partial inverses** for T. For $v = (v_1, \ldots, v_i) \in \mathcal{T}$ define $T_v(x) = T_{v_i}(...(T_{v_1}(x)))$. Also define the tree name of x by $\mathcal{T}_x(v) = K(T_v(x))$. More generally for any finite set P we call a function h from $\mathcal{T}$ to P a $\mathcal{T}, P$ name. For us a **subtree** of $\mathcal{T}$ will be a path connected set of vertices. Notice that this means a subtree will have a **root** which is the unique vertex in it of least depth, and will consist of a collection of connected paths descending from this root. If $\mathcal{T}'$ is any subtree of $\mathcal{T}$ then a $\mathcal{T}', P$ name h is any function from $\mathcal{T}'$ to P. A $\mathcal{T}', P$ name on a subtree gives rise to a collection of names indexed by intervals in $-\mathbb{N}$ by listing in negative order the names that appear along vertices of the subtree (with multiplicities). Be sure to keep in mind that in this translation vertices at depth n in the tree correspond to point in T^{-n} for the action, i.e. there is a switch in sign. More accurately, such a name on a subtree gives rise to a measure or distribution on such finite names where each name of length t is given mass 2^{-t} (again counting multiplicities). If this original tree name is the tree name of a point, then this distribution will be the conditional distribution of the various past cylinders given the path to the root of the subtree.

We say that a vertex v is in the **bottom** of the subtree T' if no extension of v is a vertex in T'. We define T''', the **concatenation** of two subtrees T' and T'', as follows. Let

$$T''' = T' \cup \left(\bigcup_{\text{bottom of } \mathcal{T}'} vT'' \right)$$

where the second union is taken over all v which are in the bottom of T'. That is to say we attach to each vertex in the bottom of T a copy of the subtree T'. We concatenate tree names in an analogous manner by extending the labeling of $\mathcal{T}$ to be the labeling of $\mathcal{T}'$ on each of the copies of $\mathcal{T}'$ attached at the bottom of $\mathcal{T}$.

Let $\mathcal{A}$ be the collection of all bijections of the vertices of $\mathcal{T}$ that preserve the tree structure. We refer to this as the group of **tree automorphisms**. Let $\mathcal{A}_n$ be the bijections of the vertices of $\mathcal{T}_n$ preserving the tree structure. To give a representation to such automorphisms A notice that from A we obtain a permutation π_v of $\{0, 1\}$ at each vertex giving the rearrangement of its 2 immediate extensions. An automorphism of $\mathcal{T}_n$ will be represented by an assignment of a permutation of $\{0, 1\}$ to each vertex of the tree including the root and excepting those at depth n.

Fix a partition P. The Hamming metric between two $\mathcal{T}_n, P$ names W and W' is given by

$$\bar{d}_n(W, W') = \frac{\#\text{ of } v \in \mathcal{T}_n \setminus \mathcal{T}_{n-1} \text{ such that } W(v) \neq W'(v)}{2^n}.$$

Now define

$$\bar{v}_n(y, y') = \bar{v}_n^P(y, y') = \inf_{A \in \mathcal{A}_n} \left(\bar{d}_n(A(\mathcal{T}_y), \mathcal{T}_{y'}) \right).$$

In the case that $\{\mathcal{F}_n\}$ comes from a dyadic endomorphism Vershik's standardness criterion is the following.

Theorem 2.1 *[8] $\{\mathcal{F}_n\}$ is standard iff for every finite partition P,*

$$\int \bar{v}_n^P(y, y')d(\nu \times \nu) \to 0.$$

Remark 2.1 *A proof of this can also be found in [4].*

3 Construction

The construction will be done by cutting and stacking. Cutting and stacking in $\mathbb{Z}$ can be viewed in two ways. One can regard the construction as building a sequence of Rokhlin towers of intervals labeled by symbols from some labeling set P. Successive towers are built by slicing up and restacking. The map is defined on ever larger parts of the space until it is eventually defined almost everywhere. One can also view the stack as a distribution on the set of all finite names (most of course given mass zero). For each length $k \in \mathbb{N}$ one can construct a measure on cylinders of length k from each stack by calculating the density of occurrence of that cylinder within the stack. These measures on cylinders will converge weak* to a shift invariant measure on $P^{\mathbb{Z}}$. The constructed action then is the shift map on $P^{\mathbb{Z}}$. Usually both these views give the same action although this depends on whether the labels in the first description give a generating partition for the action. For our construction we will follow the latter perspective by constructing names on finite subtrees. We have already described how to translate such a name into a distribution on names on intervals in $-\mathbb{N}$. This translation links our work to the traditional cutting and stacking construction of $\mathbb{Z}$ actions.

The construction will build inductively one $\mathcal{T}_{H(n)}$ name, B_n for each n. From this sequence of names we will construct a sequence of measures on $\mathcal{T}_k, P$-names by calculating the density of occurrences of the subtree name within each B_n. These measure will converge weak* to a measure on $\mathcal{T}, P$ names. This measure extends to a shift invariant measure on $P^{\mathbb{Z}}$ and its restriction to $P^{\mathbb{N}}$ will be the endomorphism we are interested in. Disjoint occurrences of copies of the name B_n in the past trees of points will place a block structure on these tree names. We consider two points x and y and their 2^m inverse images under (T^{-m}). The construction will be done in such a way that it will be impossible to find a pairing of the 2^m inverse images of x with those of y by a tree automorphism that will match up the block structures of the paired inverse images. But there is a bijection of the inverse images which does not preserve the tree structure and which matches up the block structure.

To do the construction we will need three sequences of integers, $H(n)$, the height of n tree, $N(n)$, the number of copies of $n-1$ trees concatenated to form the n tree, and a parameter $F(n)$. These sequences will be defined inductively. Let $H(1)=1000$. Given that $H(n-1)$ has been defined choose $F(n)$ so that $\sqrt{F(n)} > 2^{n+100}H(n-1)$. Also choose $N(n)$ so that $H(n)=3F(n)+N(n)H(n-1) \geq 2^{n+100}F(n)$.

An element of the partition P is of the form (a,n,v), where $a \in \{0,1\}$, $n \in \mathbb{N}$, and $v \in \mathcal{T}$. Notice that P will not be a finite partition. Both standardness and Bernoullicity are characterized by the behavior of finite partitions. We will explain how this issue is handled at the appropriate points.

For any $v \in \mathcal{T}$ define

$$f_n(v) = \text{minimum } \{3F(n), \text{ the smallest } k \leq |v| \text{ such that } \textstyle\sum_1^k v_i = F(n)\}.$$

We will now inductively define $\mathcal{T}_{H(n)}, P$ names which we call B_n. The name B_1, is defined so that each vector $v \in \mathcal{T}_{H(1)}$ gets a distinct label. For any $v \in \mathcal{T}_{H(1)}$ assign $B_1(v) = (v_{|v|}, 1, v)$.

Now assume that B_n has been defined. Create the subtree that consists of all vectors $v \in \mathcal{T}_{3F(n)}$ such that $\sum_1^{|v|} v_i \leq F(n)$. Give each of these vertices a label in P which is not seen in B_n. Now concatenate this tree name with $N(n)$ copies of B_{n-1}. Then for any vertex $v \in \mathcal{T}_{H(n)}$ which has not yet received a label assign it a label which has not been used before.

To make this precise for any $v \in \mathcal{T}_{H(n)}$ such that $\sum_1^{|v|} v_i < F(n)$ or $|v| > f_n(v) + N(n)H(n-1)$ assign $B_n(v) = (v_{|v|}, n, v)$. If $v \in \mathcal{T}_{H(n)}$ such that $|v| - f_n(v) \in [1, N(n)H(n-1)]$ let

$$\hat{v}_i = v_{i+f_n(v)+\lfloor (|v|-f_n(v))/H(n-1) \rfloor H(n-1)},$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Then define $B_n(v) = B_{n-1}(\hat{v})$. This inductively defines B_n.

The $\mathcal{T}_{H(n)}$ name B_n defines a measure μ_n on $P^{\mathcal{T}_k}$, $k \leq H(n)$ as follows. Any $h \in P^{\mathcal{T}_k}$ receives mass

$$\mu_n(h) = \sum \frac{1}{(H(n)-k+1)2^{|v|}}$$

where the sum is taken over all $v \in \mathcal{T}_{H(n)-k}$ such that $h(v') = B_n(vv')$ for all $v' \in \mathcal{T}_{H(n-1)}$. The measures μ_n, which project, as we have described, to measures on names labeled by $[-n,\ldots,-1]$ we still refer to as μ_n. As these measures on names are precisely what would arise if one did traditional cutting a stacking to create the distribution on names associated with B_n we conclude the μ_n converge in the weak * topology to a shift invariant measure $\hat{\mu}$ on $P^{\mathbb{Z}}$. Restrict $\hat{\mu}$ to $P^{\mathbb{N}}$ to give the endomorphism T we claim is Bernoulli but not standard.

As the labels used to *fill in* the top and bottom of the tree name only appear there, the block structure on the past trees of points are unique.

Let K_0 be the set of $x \in X$ such that $P(x)$ is of the form $(0, *, *)$ and K_1 be the set of $x \in X$ such that $P(x)$ is of the form $(1, *, *)$. One sees from the construction that $T : K_0 \to X$ and $T : K_1 \to X$ are both 1-1 and onto. This defines partial inverses T_0^{-1} and T_1^{-1} both of which have constant R.N. derivatives of $1/2$ and hence T is a uniformly dyadic endomorphism. By the method described in the previous section we can define T_v for any $v \in \mathcal{T}$ and $\mathcal{T}_x$ for any $x \in X$.

We say that a point $x \in X$ is **in the** n **block** if there exists $v_x \in \mathcal{T}_{H(n)}$ such that for all $v' \in \mathcal{T}_{H(n)-|v_x|}$ we have

$$\mathcal{T}_x(v') = B_n(v_x v').$$

We say that x is in the **top of the** n **block** if $|v_x| = 0$. For general tree names we will use the corresponding definitions of being in the n block or being in the top of the n block.

Lemma 3.1 *For any $n \geq 2$ and $k \in [0, H(n-1))$ and $l \geq 3F(n)$ the number of $v \in \mathcal{T}_l \setminus \mathcal{T}_{l-1}$ such that $f(v') = k \bmod H(n-1)$ is less than $2^{l+1}/H(n-1)$.*

Proof: It causes no loss of generality to assume that $l = 3F(n)$. Since $\sqrt{F(n)} >> H(n-1)$ this follows from the local central limit theorem. See, for example, [2] page ??.

□

A slightly different version of this lemma is the following.

Corollary 3.1

$$\sup_k 2^{-k} \text{ (\# of } \tilde{v} \text{ such that } |\tilde{v}| = k \text{ and } T_{\tilde{v}} B_n \text{ is in the top of the } n-1 \text{ block)} \leq 2/H(n-1).$$

Proof: It causes no loss of generality to assume that $k \geq 3F(n)$. This is because if $k \leq 3F(n)$ then the quantity we are trying to maximize is greater for $k + H(n-1)$ than for k. Then this is just a restatement of the previous lemma. □

Lemma 3.2 *The endomorphism (X, T, μ) has entropy log 2.*

Proof: By definition P is a generating partition for the endomorphism. Thus the entropy of the endomorphism is the same as the entropy of the endomorphism relative to P.

For almost every point $x \in X$ and integer k there is an n such that x is in the n block and $H_n - |v_n(x)| > k$. As there is only one n block and x is in the n block, conditioning on $P(x), P(T^1(x)), ..., P(T^{H_n}(x))$ determines v_x. Thus, as there is only one n block, conditioning on the sequence $P(x), P(T^1(x)), ..., P(T^{H_n}(x)), ...$ there are 2^k possibilities for $P(T^{-1}(x)), ..., P(T^{-k}(x))$ and they are all equally likely. Thus the entropy of the endomorphism is log 2. $\square$

4 The sequence of σ-algebras is not standard

Let $\epsilon_1 = 1$ and $\epsilon_n = \epsilon_{n-1}(1 - 2^{-n-95})$. Choose $\epsilon = \lim \epsilon_n > 0$. The main part of the proof that (X, T, μ) does not generate a standard decreasing sequence of σ-algebras is the following inductive statement.

Lemma 4.1 *Given any $n \in \mathbb{N}$, $v \in \mathcal{T}_{H(n)} \setminus \mathcal{T}_{3F(n)}$, and j, $0 < j \leq H(N) - |v|$, we have*

$$\bar{v}_j(T_v B_n, B_n) > \epsilon_n.$$

Before we start the proof of this lemma we will sketch the proof and introduce some notation. We argue by induction in n. The main idea is to break up the sum in the calculation of $\bar{v}_j(T_v B_n, B_n)$ into the weighted average of terms of the form $\bar{v}_k(T_{v'} B_{n-1}, B_{n-1})$. The variation in the value of f_n will ensure that for most of the terms being averaged $|v'| > 3F(n-1)$. Arguing inductively in n we will bound $\bar{v}_j(T_v B_n, B_n)$ in terms of values $\bar{v}_k(T_{v'} B_{n-1}, B_{n-1})$. Now we introduce notation to make this precise.

Given $n \in \mathbb{N}$, $v \in \mathcal{T}_{H(n)}$, $j \in \mathbb{N}$ such that $0 < j \leq H(n) - |v|$, and an automorphism $A \in \mathcal{A}_j$ we will define a few subsets of $\mathcal{T}_j$. First let V_1 be all $\tilde{v} \in \mathcal{T}_j \setminus \mathcal{T}_{j-1}$ such that $T_{v\tilde{v}} B_n$ is not in the $n-1$ block or $T_{A(\tilde{v})} B_n$ is not in the $n-1$ block.

Let V_2 be all $\tilde{v} \in \mathcal{T}_j$ such that

1. either $T_{v\tilde{v}} B_n$ or $T_{A(\tilde{v})} B_n$ is in the top of an $n-1$ block,
2. no extension of $\tilde{v}$ is in V_1,
3. there is no $v'' \in \mathcal{T}_j$ such that v'' is an extension of v and $T_{v\tilde{v}''} B_n$ is in the top of an $n-1$ block, and
4. there is no $v'' \in \mathcal{T}_j$ such that v'' is an extension of $A(\tilde{v})$ and $T_{v''} B_n$ is in the top of an $n-1$ block.

Now for each $v \in \mathcal{T}_j \setminus \mathcal{T}_{j-1}$ either v is in V_1 or v has exactly one contraction in V_2. But both can not happen. From this it is easy to verify that

$$\frac{1}{2^j}(\#\text{ of }\tilde{v} \in V_1) + \sum_{\tilde{v} \in V_2} 2^{-|\tilde{v}|} = 1 \tag{1}$$

and

$$\begin{aligned} d_j(T_v B_n, A(B_n)) &= \frac{1}{2^j}(\#\text{ of }\tilde{v} \in V_1 \text{ such that } P(T_{v\tilde{v}} B_n) \neq P(T_{v' A(\tilde{v})} B_n)) + \\ & \sum_{\tilde{v} \in V_2} 2^{-|\tilde{v}|} d_{j-|\tilde{v}|}(T_{v\tilde{v}} B_n, A_{\tilde{v}}(T_{v' A(\tilde{v})} B_n)). \end{aligned} \tag{2}$$

We have used the notation $A_{\tilde{v}}$ to denote the restriction of A to $\tilde{v}\mathcal{T}_{n-|\tilde{v}|}$.

Since one of $T_{v\tilde{v}} B_n$ or $T_{v' A(\tilde{v})} B_n$ is in the top of an $n-1$ block we can almost use the induction hypothesis to get a bound on the summands in line 2. Suppose it is $T_{v\tilde{v}} B_n$ that is in the top of an $n-1$ block. In order to apply the induction hypothesis we just need to make sure that $T_{v' A(\tilde{v})} B_n$ is not in the top $3F(n-1)$ levels of the $n-1$ block.

Now we define sets V_3 and V_4 so that V_2 is the disjoint union of V_3, where the induction hypothesis applies, and V_4, where it does not. Let h be the largest $k \leq j$ such that $|v| + k - f_n(v) = 0 \bmod H(n-1)$. Let V_3 consist of all $\tilde{v} \in V_2$ such that

1. $T_{v\tilde{v}} B_n$ is in the top of the $n-1$ block and $(h - f_n(A(\tilde{v})) \bmod H(n-1)) > 3F(n-1)$ or

2. $T_{A(\tilde{v})} B_n$ is in the top of the $n-1$ block and $|A(\tilde{v})| - h > 3F(n-1)$

Let $V_4 = V_2 \setminus V_3$.

Lemma 4.2 *Given n and let $v \in \mathcal{T}_{H(n)} \setminus \mathcal{T}_{3F(n)}$. Then for any $j \leq H(n) - |v|$ we have*

$$\frac{1}{2^j}(\#\textit{ of }\tilde{v} \in V_1) + \sum_{\tilde{v} \in V_3} 2^{-|\tilde{v}|} > 1 - 2^{-n-95}.$$

Proof: By line 1 this is equivalent to showing that

$$\sum_{\tilde{v} \in V_4} 2^{-|\tilde{v}|} < 2^{-n-95}.$$

If $T_{v\tilde{v}} B_n$ is in the top of the $n-1$ block then $|\tilde{v}| = h$. The number of $\tilde{v}$ with $|\tilde{v}| = h$ and $(h - f_n(A(\tilde{v})) \bmod H(n-1)) \leq 3F(n-1)$ is

$$
\begin{aligned}
&\leq (3F(n-1)+1)\sup_k\{\#\text{ of } v' \in \mathcal{T}_j \setminus \mathcal{T}_{j-1} \text{ such that } f_n(\tilde{v}) = k \bmod H(n-1)\} \\
&\leq 4F(n-1)2^h\frac{2}{H(n-1)} \\
&\leq 2^h\frac{8F(n-1)}{H(n-1)}.
\end{aligned}
$$

The sum $\sum 2^{-|A(\tilde{v})|}$ over all $\tilde{v}$ such that $T_{A(\tilde{v})}B_n$ is in the top of the $n-1$ block and $0 \leq |A(\tilde{v})| - h \leq 3F(n-1)$ is

$$
\begin{aligned}
&\leq (3F(n-1)+1)\sup_k 2^{-k}\ (\#\text{ of } \tilde{v} \text{ such that } |\tilde{v}| = k \text{ and } T_{\tilde{v}}B_n \text{ is in the top of the } n-1 \text{ block}) \\
&\leq 4F(n-1)\frac{2}{H(n-1)} \\
&\leq \frac{8F(n-1)}{H(n-1)}.
\end{aligned}
$$

Thus combining these two estimates gives

$$
\begin{aligned}
\sum_{\tilde{v}\in V_4} 2^{-|\tilde{v}|} &\leq 2^{-h}2^h\frac{8F(n-1)}{H(n-1)} + \frac{8F(n-1)}{H(n-1)} \\
&\leq (16)2^{-n-99} \\
&\leq 2^{-n-95}.
\end{aligned}
$$

$\square$

Proof of Lemma 4.1: The base case is trivial. This is because if $v \neq v'$ then $B_1(v) \neq B_1(v')$.

For any automorphism A

$$\begin{aligned}
d_j(T_vB_n, A(B_n)) &= \frac{1}{2^j}(\#\text{ of } \tilde{v} \in V_1 \text{ such that } P(T_{v\tilde{v}}B_n) \neq P(T_{v'A(\tilde{v})}B_n)) \\
&\quad \sum_{\tilde{v}\in V_3} 2^{-|\tilde{v}|} d_{j-|\tilde{v}|}(T_{v\tilde{v}}B_n, A_{\tilde{v}}(T_{v'A(\tilde{v})}B_n)) + \\
&\quad \sum_{\tilde{v}\in V_4} 2^{-|\tilde{v}|} d_{j-|\tilde{v}|}(T_{v\tilde{v}}B_n, A_{\tilde{v}}(T_{v'A(\tilde{v})}B_n)) \\
&\geq \frac{1}{2^j}(\#\text{ of } \tilde{v} \in V_1 \text{ such that } P(T_{v\tilde{v}}B_n) \neq P(T_{v'A(\tilde{v})}B_n)) \\
&\quad \sum_{\tilde{v}\in V_3} 2^{-|\tilde{v}|} d_{j-|\tilde{v}|}(T_{v\tilde{v}}B_n, A_{\tilde{v}}(T_{v'A(\tilde{v})}B_n)) \\
&\geq \frac{1}{2^j}(\#\text{ of } \tilde{v} \in V_1) + \sum_{\tilde{v}\in V_3} 2^{-|\tilde{v}|} d_{j-|\tilde{v}|}(T_{v\tilde{v}}B_n, A_{\tilde{v}}(T_{v'A(\tilde{v})}B_n)) \qquad (3) \\
&\geq \frac{1}{2^j}(\#\text{ of } \tilde{v} \in V_1) + \sum_{\tilde{v}\in V_3} 2^{-|\tilde{v}|} \inf_{3F(n-1)<|v''|\leq H(n-1)} \bar{v}_{j-|\tilde{v}|}(B_{n-1}, T_{v''}B_{n-1}) (4) \\
&\geq \epsilon_{n-1}\frac{1}{2^j}(\#\text{ of } \tilde{v} \in V_1) + \epsilon_{n-1}\sum_{\tilde{v}\in V_3} 2^{-|\tilde{v}|} \\
&> \epsilon_{n-1}(1-2^{-n-95}) \\
&> \epsilon_n.
\end{aligned}$$

Line 3 is true because if $\tilde{v} \in V_1$ then either $T_{v\tilde{v}}B_n$ is not in the $n-1$ block or $T_{A(\tilde{v})}B_n$ is not in the $n-1$ block. Since $v\tilde{v} \neq A(\tilde{v})$ $B_n(v\tilde{v}) \neq B_n(A(\tilde{v}))$.

Line 4 is true because one of $T_{v\tilde{v}}B_n$ or $T_{A(\tilde{v})}B_n$ is in top of the $n-1$ block by the definition of V_2. The induction hypothesis applies because of the definition of V_3. As the above calculation is independent of A we have a bound on $\bar{v}$. □

Now we are ready to prove that $\int \bar{v}_n^P(y, y')d(\nu \times \nu) \not\to 0$.

Lemma 4.3 *For all n there exists X_n and Y_n with $\mu(X_n), \mu(Y_n) > 1/5$ with the following property. For any $x \in X_n$ and any $y \in Y_n$*

$$\bar{v}_{H(n-1)}(\mathcal{T}_x, \mathcal{T}_y) \geq \epsilon.$$

Proof: If a point x is in the n block then we get a vertex v_x. Define

$$X_n = \{x \mid |v_x| - f_n(v_x) \bmod H(n-1) \in (H(n-1)/8, 3H(n-1)/8)\}.$$

Define

$$Y_n = \{y \mid |v_y| - f_n(v_y) \bmod H(n-1) \in (5H(n-1)/8, 7H(n-1)/8)\}.$$

Given $x \in X_n$ and $y \in Y_n$ let $k = H(n-1) - [|v_x| - f_n(v_x) \bmod H(n-1)]$.

Now

$$\bar{v}_{H(n-1)}(\mathcal{T}_x, \mathcal{T}_y) \geq \inf_{A \in \mathcal{A}_k} \frac{1}{2^k} \sum_{|\tilde{v}|=k} \bar{v}_{H(n-1)-k}(T_{\tilde{v}}(\mathcal{T}_x), T_{A(\tilde{v})}(\mathcal{T}_y)).$$

By the choice of k all the $\bar{v}$ terms are of the form $\bar{v}_{H(n-1)-k}(B_{n-1}, T_{v''}B_{n-1})$ with $v'' \in \mathcal{T}_{H(n-1)} \setminus \mathcal{T}_{H(n-1)/4}$. Thus

$$\bar{v}_{H(n-1)}(\mathcal{T}_x, \mathcal{T}_y) \geq \inf \bar{v}_{H(n-1)-k}(B_{n-1}, T_{v''}B_{n-1}) \geq \epsilon,$$

where the inf is taken over all $v'' \in \mathcal{T}_{H(n-1)} \setminus \mathcal{T}_{H(n-1)/4}$. The last inequality is by lemma 4.3. By the definition of $F(n)$ and $H(n)$ we get

$$\mu(X_n) = \mu(Y_n) \geq \frac{1}{4}\mu(B_{n-1}) \geq \frac{1}{4} \prod_{j \geq n} \frac{3F(j)}{H(j)} \geq \frac{1}{5}.$$

which proves the lemma. $\square$

Theorem 4.1 (X, T, μ) *does not generate a standard decreasing sequence of* σ*-algebras.*

Proof: From lemma 4.3 it follows that for all n

$$\int v^P_{H(n)}(y, y')d(\nu \times \nu) > \epsilon/25.$$

Thus

$$\int v^P_j(y, y')d(\nu \times \nu) \not\to 0.$$

Now choose a finite partition P' which agrees with P on all but $\epsilon/100$ of the space. Then it is clear that for all n

$$\int v^{P'}_{H(n)}(y, y')d(\nu \times \nu) > \epsilon/50$$

and

$$\int v^{P'}_j(y, y')d(\nu \times \nu) \not\to 0.$$

Thus by theorem 2.1 (X, T, μ) does not generate a standard decreasing sequence of σ-algebras. $\square$

5 The two sided extension is Bernoulli

This is proven by showing that (T, P) is v.w.B. Of course v.w.B. is a condition on finite partitions but if one verifies it for a countable partition it still implies Bernoullicity. We will use the same techniques used by Ornstein in [7]. For any $v = (v_1, ..., v_k)$ and any $i \leq k$ let $v|_i = (v_1, ..., v_i)$. Also let $v|^i = (v_{i+1}, ..., v_{|v|})$. Thus $v = v|_i v|^i$. For a fixed n and any $v_1 \in \mathcal{T}$ let S_{v_1} be all extensions v' of v_1 such that $|v'| = |v_1| + l_n$ where l_n is a number defined below. The crux of the proof is the following matching lemma.

Lemma 5.1 *For all n and $k \leq n$ there exists $V \subset \mathcal{T}_{H_n}$ and $l_n \in N$ with the following property. Then for any $v_1, v_2 \in V$ there exists a one to one map $M : S_{v_1} \to S_{v_2}$ such that*

$$\begin{aligned}\textstyle\sum_{v \in S_v} \{\#\ \textit{of}\ i\ \textit{such that}\quad & T_{v|_{|v_1|+i}} B_n \textit{ is in the top of the } n-k \textit{ block and} \\ & T_{M(v)|_{|v_2|+i}} B_n \textit{ is in the top of the } n-k \textit{ block}\} \\ & \geq 2^{l_n} \frac{l_n - 2H(n-1)}{H(n-k)} \mu(B_{n-k})(1 - (9/10)^{k-1}).\end{aligned}$$

Proof: Fix n and the proof is by induction on k. Let $V = \mathcal{T}_{H(n)-l_n-3F(n)} \setminus \mathcal{T}_{3F(n)}$ where $l_n = H(n)/2^n$. For $k = 1$ the statement is vacuously true.

Note that by the previous section M cannot preserve the tree structure. Along with any $v \in S_{v_1}$ there is a corresponding sequence in P^{l_n}. It is defined by $n_v = (B_n(v|_{|v_1|+i}), ..., B_n(v|_{|v_1|+l_n}))$. For a given $v \in S_{v_1}$ we say that the j blocks are the intervals of the form $[i, i + h(j))$ which are contained in $[1, l_n]$ and $T_{v|_{|v_1|+1}}$ is in the top of the j block. It causes no loss of generality to assume that the extensions of v_1 and v_2 have the same number $n - 1$ blocks. We will show we can choose M to have the following property. If the sequences corresponding to two vertices in S_{v_1} disagree only inside $n-k$ blocks in $n-1$ blocks then M applied to these vertices yields two vertices whose corresponding sequences differ inside $n - k$ blocks inside of $n-1$ blocks. (i.e. If $(n_v)_i = (n_{v'})_i$ for all i inside $n-k$ blocks inside $n-1$ blocks of v then $(n_{M(v)})_i = (n_{M(v')})_i$ for all i inside $n - k$ blocks inside $n - 1$ blocks of $M(v)$.)

Consider $v \in S_{v_1}$ and all other v' such that $(n_v)_i = (n_{v'})_i$ for all i inside $n - k$ blocks inside $n-1$ blocks of v. We now describe how to modify M on this set. When we apply this procedure to all such sets we get M' such that the induction hypothesis holds for $k + 1$.

Now consider the $n-k$ blocks of v that are not the same as some $n-k$ block of $M(v)$. Pair these with the $n - k$ blocks of $M(v)$ that are not the same as some $n - k$ block of v in such a way that the overlap of paired blocks is at least 1/3 of the length of these blocks.

Now pick one pair of $n-k$ blocks. Say one of them is $[i, i+H(n-k))$ and the other is $[j, j+H(n-k))$. Choose M' so that the number of v' in this set with

$$(i + f_{n-k}(v'|^i)) - (j + f_{n-k}(M'(v')|^j)) = 0 \bmod H(n-1)$$

is maximized. This can be done for at least half of the v' in the set since since $\sqrt{F(n-k)} >> H(n-k-1)$. Now repeat this procedure for the other paired $n-k$ blocks. Then repeat this procedure for another v. Doing this we have matched at least $1/10$ of the $n-k-1$ blocks inside the unmatched $n-k$ blocks which justifies the induction hypothesis for $k+1$. □

Theorem 5.1 *The transformation $(\bar{X}, T, \bar{\mu})$ is Bernoulli.*

Proof: Since (X, T, μ) is dyadic and has entropy $\log 2$ we need only to show that $(X, T, \bar{\mu})$ is very weak Bernoulli. It also suffices to show that $(\bar{X}, T^{-1}, \bar{\mu})$ is very weak Bernoulli.

Given ϵ choose n and k so that

$$2^{l_n}(9/10)^{k-1} + (1 - \mu(B_{n-k})) + \frac{2H(n-1)}{l_n} < \epsilon$$

and

$$\frac{3F(n) + l_n}{H(n)}\mu(B_n) > 1 - \epsilon.$$

Let G be the set of all x such that x is in the n block and $v_x \in V_n$. Then

$$\mu(G) = \frac{3F(n) + l_n}{H(n)}\mu(B_n) > 1 - \epsilon.$$

Now given any $x, x' \in G$ we get $v_x, v_{x'} \in V_n$. Now choose M so that

1. the conclusion of the previous lemma is satisfied and
2. if $M(v) = v'$ and $[i, i+H(n-k))$ is an $n-k$ block for both v and v' then $(n_v)_j = (n_{v'})_j$ for all $j \in [i, i+H(n-k))$.

Now the fraction on $n-k$ blocks inside $n-1$ blocks that are unmatched is at most $(9/10)^{k-1}$. The fraction of an $n-1$ block that is not part of $n-k$ blocks is less than $(1-\mu(B_{n-k}))$. While the fraction of $[1, l_n]$ that is not in an $n-1$ block is at most $\frac{2H(n-1)}{l_n}$. Thus

$$\frac{1}{2^{l_n} l_n} \sum_{v \in S(v)} \# \text{ of } i =\in [1, l_n] \text{ such that } (n_v)_i \neq (n_{M(v')})_i$$

$$\leq (9/10)^{k-1} + (1 - \mu(B_{n-k})) + \frac{2H(n-1)}{l_n} < \epsilon.$$

Thus T^{-1} is very weak Bernoulli. □

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