At the start of January 1 a reservoir has 100,000 gallons of water in it.

Could the graph below represent the change in the amount of water as a function of time?
The amount of water in a reservoir is never negative (or $W(t) \geq 0$ for all $t$).

So this could be the graph of the change in water if for all $t$

$$W(t) \geq 0$$

$$W(0) + (W(t) - W(0)) \geq 0$$

$$\int_0^t W'(t) \, dt \geq -100,000$$

We can see that $W(t)$ is lowest at $t = 50$. And by estimating the area we can see that

$$\int_0^{50} W'(t) \, dt \geq -100,000.$$ 

So this could be the graph.
1. At what time does the reservoir have the most water in it?

2. Approximately how much water is in the reservoir on April 11?

   Use the midpoint rule with \( n = 5 \).

3. When is the quantity of water in the reservoir decreasing the fastest?

4. Is the water level increasing or decreasing on February 1?

5. Is the water level higher on March 1 or April 1?
1. There is the most water in the reservoir when
\[ \int_0^t W'(t) \, dt \]
is the largest. This at \( t = 10 \) or midnight January 10.

2. The start of April 11 is \( t = 100 \). We have \( \Delta t = 20 \) and
\[
W(100) = W(0) + \int_0^{100} W'(t) \, dt \\
\approx 100,000 + 20(0 + (-2) + 0 + .5 + (-.25))(1000) \\
\approx 65,000.
\]

3. At time \( t = 25 \) or January 25.

4. Since \( W'(31) \) is negative the amount of water is decreasing.
5. The change in water during March is

\[ W(90) - W(59) = \int_{59}^{90} W'(t) \, dt \]

which is positive so the amount of water increased.
Let

\[ f(x) = \int_{1}^{2x-1} t^2 + \ln(t) \, dt. \]

Find the equation of the tangent line to \( y = f(x) \) at \( x = 1 \).

By the Fundamental Theorem of Calculus

\[ f'(x) = ((2x - 1)^2 + \ln(2x - 1)) \frac{d}{dx}(2x - 1) \]
\[ = 2((2x - 1)^2 + \ln(2x - 1)). \]

So the slope of tangent line to \( y = f(x) \) at \( x = 1 \) is

\[ f'(1) = 2(1^2 + \ln(1)) = 2. \]
The tangent line goes through the point 

$$(1, f(1)).$$

We have that 

$$f(1) = \int_1^{2(1)-1} t^2 + \ln(t) \, dt = \int_1^1 t^2 + \ln(t) \, dt = 0.$$ 

The equation of the line through $(1,0)$ with slope 2 is 

$$y = 2x - 2.$$
Evaluate the following integrals

1. $\int \frac{x^3}{4x^2+1} \, dx$.

2. $\int_{-3}^{3} x^3 \cos (x^4 + 1) \, dx$

3. $\int_{e}^{4} \frac{dx}{x \sqrt{\ln x}}$
We use the substitution $u = 4x^2 + 1$. We have

$$\frac{du}{dx} = 8x \text{ or } x \ dx = \frac{du}{8}$$

and

$$x^2 = \frac{u - 1}{4}.$$

$$\int \frac{x^3}{4x^2 + 1} \ dx = \int \frac{x^2(x \ dx)}{4x^2 + 1}$$

$$= \int \frac{(u - 1)}{4} \left( \frac{du}{8} \right)$$

$$= \int \frac{u}{32u} \ du$$

$$= \int \left( 1 - \frac{1}{u} \right) \ du$$

$$= \frac{1}{32} \left( u - \ln u \right) + c$$

$$= \frac{1}{32} \left( (4x^2 + 1) - \ln(4x^2 + 1) \right) + c.$$
We use the substitution \( u = x^4 + 1 \). We have

\[
\frac{du}{dx} = 4x^3 \quad \text{or} \quad \frac{du}{4} = x^3 \, dx.
\]

When

\[
x = -3, \quad u = (-3)^4 + 1 = 82
\]

and when

\[
x = 3, \quad u = (3)^4 + 1 = 82.
\]

Thus

\[
\int_{x=-3}^{x=3} x^3 \cos x^4 + 1 \, dx = \int_{u=82}^{u=82} \cos u \frac{du}{4}
\]

\[
= \frac{1}{4} [\sin(u)]_{u=82}^{u=82}
\]

\[
= \frac{1}{4} (\sin(82) - \sin(82))
\]

\[
= 0.
\]
Or we could have noticed that

\[ (-x)^3 \cos((-x)^4 + 1) = -x^3 \cos(x^4 + 1). \]

Since the limits of integration are 3 and -3 the integral will be 0.
We use the substitution \( u = \ln x \). Thus we have
\[
\frac{du}{dx} = \frac{1}{x} \quad \text{or} \quad du = \frac{dx}{x}.
\]
When
\[
x = e, \quad u = 1
\]
and when
\[
x = e^4, \quad u = 4.
\]
Thus
\[
\int_{x=e}^{x=e^4} \frac{dx}{x\sqrt{\ln x}} = \int_{u=1}^{u=4} \frac{du}{\sqrt{u}}
\]
\[
= \int_{u=1}^{u=4} u^{-1/2} \, du
\]
\[
= \left[ 2u^{1/2} \right]_{u=1}^{u=4} \, du
\]
\[
= 2(4)^{1/2} - 2(1)^{1/2}
\]
\[
= 2.
\]
Let $R$ be the region between the lines

\[ y = x + 5, \ y = 2, \ \text{and} \ y = -1 \]

and the parabola

\[ y^2 = x. \]

1. Set up, but do not evaluate, integral(s) that represent the area of $R$, both integrating with respect to $x$ and with respect to $y$.

2. Set up, but do not evaluate, an integral for the volume of the solid $S$ formed by revolving $R$ around the line $y = 2$.

3. Set up, but do not evaluate, an integral for the volume of the solid $T$ formed by revolving $R$ around the line $x = -7$. 

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First we sketch the region.
We first set up an integral with respect to $y$.

We solve the equation of the right hand boundary in terms of $x$ to get

$$x = y^2$$

and we solve the equation of the left hand boundary in terms of $x$ to get

$$x = y - 5.$$  

So the area is

$$\int_{-1}^{2} y^2 - (y - 5) \, dy$$
To write integrals with respect to $x$ we must break up the region into four parts, $R_1$, $R_2$, $R_3$ and $R_4$.

The area of $R_1$ is
\[\int_{-6}^{-3} (x + 5) - (-1) \, dx.\]

The area of $R_2$ is
\[\int_{-3}^{0} (2) - (-1) \, dx.\]

The area of $R_3$ is
\[\int_{0}^{4} (2) - (\sqrt{x}) \, dx.\]

The area of $R_4$ is
\[\int_{0}^{1} (-\sqrt{x}) - (-1) \, dx.\]
We use shells.

Since we are rotating about a line parallel to the \( y \) axis and using shells we integrate with respect to \( y \).

The volume of \( S \) is given by

\[
\int_{-1}^{2} c(y) h(y) \, dy
\]

where \( h(y) \) is the height of a shell as a function of \( y \) and where \( c(y) \) is the circumference of a shell as a function of \( y \).

We see that

\[
h(y) = y^2 - (y-5) = y^2 - y + 5 \quad \text{and} \quad c(y) = 2\pi(2-y)
\]

Thus the integral is

\[
\int_{-1}^{2} 2\pi(2 - y)(y^2 - y + 5) \, dy.
\]
We use washers.

Since we are rotating about a line parallel to the $x$ axis and using washers we integrate with respect to $y$.

The volume of $T$ is given by

$$\int_{-1}^{2} \pi \left( (r_o(y))^2 - (r_i(y))^2 \right) \, dy$$

where $r_o(y)$ is the outer radius as a function of $y$ and $r_i(y)$ is the inner radius as a function of $y$.

We see that

$$r_o(y) = y^2 - (-7) = y^2 + 7$$

and

$$r_i(y) = (y - 5) - (-7) = y + 2$$

Thus the integral is

$$\int_{-1}^{2} \pi \left( (y^2 + 7)^2 - (y + 2)^2 \right) \, dy.$$