This week

- Homework $\#6$ due
- Read sections 3.9 and 3.10
- Worksheet $\#6$ on Tuesday

A baseball diamond is a square that is 90 feet on each side. A baseball player is running from first base to second base at a rate of 18 feet per second. How fast is the distance from home plate changing when the runner is midway from first base to second?

Let $d_1(t)$ be the distance of the runner to first base.
Let $d_{\text{home}}(t)$ be the distance of the runner to first base.
We are asked to find $d'_{\text{home}}(t_0)$ when $d_1(t_0) = 45$ and $d'_1(t_0) = 18$.
Then we have that

$$(d_1(t))^2 + 90^2 = (d_{\text{home}}(t))^2.$$

Now we implicitly differentiate

$$\frac{d}{dt}\left((d_1(t))^2 + 90^2\right) = \frac{d}{dt}\left(d_{\text{home}}(t))^2.\right)$$

$$2d_1(t)d'_1(t) = 2d_{\text{home}}(t)d'_{\text{home}}(t)$$

Plugging into the above equation we get

$$2(45)(18) = 2d_{\text{home}}(t_0)d'_{\text{home}}(t_0).$$

We first need to find $d_{\text{home}}(t_0)$.

$$(d_1(t_0))^2 + 90^2 = (d_{\text{home}}(t_0))^2.$$

$$(45)^2 + 90^2 = (d_{\text{home}}(t_0))^2.$$

$$45\sqrt{5} = d_{\text{home}}(t_0).$$

Plugging this value into our equation we get

$$2(45)(18) = 2(45\sqrt{5})d'_{\text{home}}(t_0).$$

$$d'_{\text{home}}(t_0) = \frac{2(45)(18)}{(2)45\sqrt{5}} = \frac{18}{\sqrt{5}}.$$

Strategy

1. Draw a picture.
2. Choose notation.
3. Write an equation.
4. Differentiate using the chain rule.
5. Substitute into the equation and solve.

Air is leaking out of a balloon. The balloon is always spherical and its volume is decreasing at a rate of 100 cc/min. When the balloon has a volume of 2000 cc how quickly is the radius decreasing?

Let $V(t)$ be the volume of the balloon at time t.
Let $r(t)$ be the radius of the balloon at time t.

$$V(t) = \frac{4}{3}\pi r^3(t)$$

Differentiating we get

$$V'(t) = 4\pi r^2(t) r'(t)$$

$$V(t_0) = 2000 = \frac{4}{3}\pi r^3(t_0) \quad \text{so} \quad r(t_0) = \sqrt[3]{\frac{1500}{\pi}} \approx 7.816.$$

$$-100 = 4\pi \left(\sqrt[3]{\frac{1500}{\pi}}\right)^2 r'(t)$$

$$r'(t) = \frac{-100(\pi)^{2/3}}{4\pi(1500)^{2/3}} \approx -.13026$$

Related Rates

Two variable resistors are connected in parallel. If the two resistors have resistance R_1 and R_2 then the effective resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 and R_2 are increasing at rates of $.3\Omega/\text{sec}$ and of $.2\Omega/\text{sec}$ respectively then how fast is R changing when $R_1 = 50$ and $R_2 = 80$?

$$\frac{1}{R(t)} = \frac{1}{R_1(t)} + \frac{1}{R_2(t)}.$$

$$\frac{-R'(t)}{(R(t))^2} = \frac{-R_1'(t)}{(R_1(t))^2} + \frac{-R_2'(t)}{(R_2(t))^2}.$$

Let t_0 be the time that we are interested in. Then we have $R_1'(t_0) = .3$ and $R_2'(t_0) = .2$ $R_1(t_0) = 50$ and $R_2(t_0) = 80$.

$$\frac{-R'(t_0)}{(R(t_0))^2} = \frac{-.3}{50^2} + \frac{-.2}{80^2}.$$

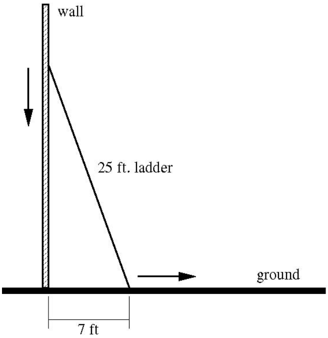
We need to find

$$R(t_0) = \frac{1}{R_1(t_0)} + \frac{1}{R_2(t_0)} = \frac{1}{50} + \frac{1}{80} = \frac{13}{400}.$$

$$\frac{-R'(t_0)}{(13/400)^2} = \frac{-.3}{50^2} + \frac{-.2}{80^2}.$$

$$R'(t_0) = -\frac{400^2}{13^2}\left(\frac{-.3}{50^2} + \frac{-.2}{80^2}\right) \approx .1432.$$

A 25 feet long is leaning against the wall of a building and then it starts to slide. When the foot of the ladder is 7 feet from the wall, the foot of the ladder is sliding at a rate of 2 ft/sec, causing the top of the ladder to slide down the wall. How fast is the top of the ladder sliding down the wall at this time?



Let $x(t)$ be the distance from the bottom of the wall to where the ladder touches the floor.
Let $y(t)$ be the distance from the bottom of the wall to where the ladder touches the wall.
Then

$$x(t)^2 + y(t)^2 = 25^2.$$

Differentiating we get

$$2x(t)x'(t) + 2y(t)y'(t) = 0.$$

We are asked to find $y'(t_0)$ when $x(t_0) = 7$ and $x'(t_0) = 2$. Thus

$$y(t_0) + 7^2 = 625 \quad \text{or} \quad y(t_0)^2 = 576 \quad \text{or} \quad y(t_0) = 24.$$

Then

$$2x(t_0)x'(t_0) + 2y(t_0)y'(t_0) = 0.$$

$$2(7)(-2) + 2(24)y'(t_0) = 0 \quad \text{or} \quad y'(t_0) = -\frac{7}{12}.$$

A car and a bicycle are both driving in straight paths that are parallel to each other. When the car passes the bicyclist they are five feet apart. The bike is traveling at a rate of 20 feet per second. When the (center of mass of the) car is 12 feet ahead of the (center of mass of the) bike, the distance between the car and the bike is increasing at a rate of 30 feet per second. How fast is the car driving?

Let $x(t)$ be the distance in the direction of travel between the bike and the car.
Let $d(t)$ be the distance between the bike and the car.
The speed of the car is $12 + x'(t)$.
Then

$$x(t)^2 + 5^2 = d(t)^2.$$

At the time we are interested in

$$12^2 + 5^2 = d(t_0)^2 d(t_0) = \sqrt{169} = 13.$$

$$2x(t_0)x'(t_0) = 2d(t_0)d'(t_0).$$

$$2(12)x'(t_0) = 2(13)(30).$$

So $x'(t_0) = 30\frac{13}{12}$ and the car is going $12 + 30\frac{13}{12} = 44.5$ feet per second.

Water is being poured into conical tank at a rate of .1 cubic meters per minute. The tank is 3 meters high and the top of the tank has a radius of 5 meters. How high is the water level when the level is rising at a rate of .03 meters per minute?

Let $r(t)$ be the radius at time t.
Let $h(t)$ be the height at time t.
Let $V(t)$ be the volume at time t.

$$V(t) = \frac{1}{3}\text{ (Base Area) (Height)} = \frac{1}{3}(\pi r(t)^2)h(t).$$

$$r(t) = \frac{5}{3}h(t).$$

$$V(t) = \frac{1}{3}\pi r(t)^2 h(t) = \frac{1}{3}\pi\left(\frac{5}{3}h(t)\right)^2 h(t) = \frac{25\pi}{27}h(t)^3.$$

Differentiating we get

$$V'(t) = \frac{25\pi}{27} 3h^2(t)h'(t).$$

$$.1 = \frac{25\pi}{27} 3h^2(t)(.03)$$

$$\frac{(.1)27}{25(3)(.03)\pi} = h^2(t).$$

$$h(t) = \sqrt{\frac{(.1)27}{25(3)(.03)\pi}} \approx 0.61804.$$

Strategy

1. Draw a picture
2. Choose notation
3. Write an equation
4. Differentiate using the chain rule
5. Substitute into the equation and solve

A baseball diamond is a square that is 90 feet on each side. One player runs from first to second and another runs from second to third. The one running from first is running at a rate of 18 feet per second while the one running from second is running at a rate of 20 feet per second. How fast is the distance between the two runners changing when the first is 30 feet from first base and the second is 40 feet from second?

Let $x(t)$ be the distance from the first runner and second base.
Let $y(t)$ be the distance from the second runner and second base.
Let $d(t)$ be the distance between the two runners.
Then we have the equation

$$d(t)^2 = x(t)^2 + y(t)^2.$$

$$2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t).$$

Let t_0 be the time when the first is 30 feet from first base and the second is 40 feet from second. We are asked to find $d'(t_0)$. We have that

$$x(t_0) = 60 \quad \text{and} \quad y(t_0) = 40.$$

Thus

$$d(t_0) = \sqrt{x(t_0)^2 + y(t_0)^2} = \sqrt{(60)^2 + (40)^2} = 20\sqrt{13}.$$

We are also given that $x'(t_0) = -18$ and $y'(t_0) = 20$.

Plugging all of this into

$$2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

we get

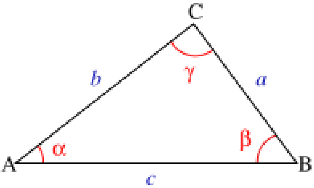
$$2(20\sqrt{13})d'(t_0) = 2(60)(-18) + 2(40)(20).$$

Solving we get

$$d'(t_0) = \frac{2(60)(-18) + 2(40)(20)}{2(20\sqrt{13})} = \frac{-14}{\sqrt{13}} \approx -3.883.$$

Sam is running counterclockwise around a circular track of radius 50 meters at a constant rate of 4 meters per second. Jane is standing 200 meters southeast of the center of the circle. How fast is the distance between Sam and Jane changing when Sam is at the east most point on the track? at the north most point on the track?

Law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos(\gamma\ ),$$

Let $d(t)$ be the distance between Sam and Jane.
Let $\gamma(t)$ be the angle between the line segments going from the center to Sam and from the center to Jane.
Then by the law of cosines we have

$$d(t)^2 = 200^2 + 50^2 - 2(200)(50)\cos(\gamma(t)).$$

$$\frac{d}{dt}\left(d(t)^2\right) = \frac{d}{dt}\left(200^2 + 50^2 - 2(200)(50)\cos(\gamma(t))\right).$$

$$2d(t)d'(t) = -2(200)(50)(-\sin(\gamma(t))\gamma'(t)).$$

To find $d'(t_0)$ we must find $d(t_0)$, $\gamma(t_0)$ and $\gamma'(t_0)$

First we find $\gamma(t_0)$ and $\gamma'(t_0)$. The length of the track is $2\pi r = 100\pi$. As Sam is running at a rate of 4 meters per second, he is does a complete lap in 25π seconds. In this time γ increases by 2π so $\gamma'(t_0) == 2\pi/25\pi = 2/25$.
From the picture we have that at the east most point $\gamma(t_0) = \pi/4$. Thus

$$d(t_0)^2 = 200^2+50^2-2(200)(50)\cos(\gamma(t_0)) = 42500-20000\cos(\pi/4)$$

$$= 42500 - 20000(\sqrt{2}/2) = 42500 - 10000\sqrt{2}.$$

$$d(t_0) = \sqrt{42500 - 10000\sqrt{2}}.$$

When Sam is at the east most point of the track

$$2d(t_0)d'(t_0) = 2(200)(50)(\sin(\gamma(t_0))\gamma'(t_0)).$$

$$2\sqrt{42500 - 10000\sqrt{2}}d'(t_0) = 2(200)(50)(\sin(\pi/4)(2/25)).$$

$$d'(t_0) = \frac{2(200)(50)((\sqrt{2}/2)(2/25))}{2\sqrt{42500 - 10000\sqrt{2}}}.$$

When Sam is at the north most point of the track $\gamma(t_1) = 3\pi/4$

$$d(t_1)^2 = 200^2 + 50^2 - 2(200)(50)\cos(\gamma(t_1)) = 42500 - 20000\cos(3\pi/4$$

$$= 42500 - 20000(-\sqrt{2}/2) = 42500 + 10000\sqrt{2}.$$

$$d(t_1) = \sqrt{42500 + 10000\sqrt{2}}.$$

$$2d(t_1)d'(t_1) = 2(200)(50)(\sin(\gamma(t_1))\gamma'(t_1)).$$

$$2\sqrt{42500 + 10000\sqrt{2}}d'(t_0) = 2(200)(50)(\sin(3\pi/4)(2/25)).$$

$$d'(t_0) = \frac{2(200)(50)((\sqrt{2}/2)(2/25))}{2\sqrt{42500 + 10000\sqrt{2}}}.$$

A street light is 15 feet high. A 6 foot tall woman is walking straight away from the streetlight at a rate of 5 feet per second. At what rate is the woman's shadow moving when she is 40 feet from the streetlight?

Let $x(t)$ be the distance from the woman to the streetlight.
Let $y(t)$ be the distance from the woman's shadow to the base of the streetlight.
By similar triangles we have the equation

$$\frac{y(t)}{15} = \frac{y(t) - x(t)}{6}.$$

$$6y(t) = 15y(t) - 15x(t).$$

$$15x(t) = 9y(t).$$

Taking derivatives we have

$$15x'(t) = 9y'(t).$$

From the problem we are given that $x'(t_0) = 5$. Thus

$$y'(t_0) = \frac{15x'(t_0)}{9} = \frac{75}{9}.$$

Linear Approximation

Recall the definition of the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

For values of x close to a we have

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}.$$

$$f(x) \approx f(a) + f'(a)(x - a).$$

We define the linearization of f at a to be

$$L(x) = f(a) + f'(a)(x - a).$$

Find the linearization of

$$f(x) = 2x\cos(3x)$$

at $x = 4$. Use it to estimate $f(4.032)$.

$$f'(x) = 2x(-3\sin(3x)) + 2\cos(3x).$$

$$f'(4) = 2(4)(-3\sin(3(4))) + 2\cos(3(4)) \approx 14.565.$$

$$f(4.032) \approx L(4.032) = f(4) + f'(4)(.032)$$

$$\approx 6.751 + (14.565)(.032)$$

$$\approx 7.217.$$

Relative change

The relative change in f is

$$\frac{\Delta f}{f(x)}.$$

By linear approximation it is about

$$\frac{\Delta f}{f(x)} = \frac{f(x+\Delta x) - f(x)}{f(x)} \approx \frac{f'(x)(\Delta x)}{f(x)}.$$

A sphere is supposed to have a radius of 21 centimeters. It will work well if its volume is within 1 % of what it is supposed to be. What percent error is acceptable for the radius?

We have that $V = \frac{4}{3}\pi r^3$. We want

$$\frac{\Delta V}{V} < .01.$$

By linearization we have

$$.01 > \frac{\Delta V}{V} \approx \frac{V'(r)(\Delta r)}{V} = \frac{4\pi r^2(\Delta r)}{(4/3)\pi r^3} = \frac{3\Delta r}{r}$$

or

$$\frac{\Delta r}{r} < \frac{1}{300}.$$

An error of $1/3$ percent in the radius is acceptable.

The amount of blood that can flow through a vein of radius r is given by the formula

$$F = kr^4$$

for some constant k. Show that at the radius changes from r to $r + \Delta r$ the relative change in the flow F is about four times the relative change in r.

$$F(r) = kr^4.$$

The relative change in r is

$$\frac{\Delta r}{r}.$$

The relative change in f is

$$\frac{F(r+\Delta r) - F(r)}{F(r)} \approx \frac{F'(r)(\Delta r)}{F(r)} \approx \frac{4kr^3(\Delta r)}{kr^4} \approx \frac{4\Delta r}{r}$$

which is four times larger than the relative change in r.

A Ferris wheel with a radius of 20 meters is rotating at a rate of one revolution every 1.5 minutes. The lowest a seat goes is 1 meter above the ground. How fast is the rider's height changing when his seat is 33 meters above the ground and falling? About how high will the rider be 2 seconds later?

We let $y(t)$ be the rider's height.

We let $\theta(t)$ be the angle between horizontal and the line from the center to the rider.

$$y(t) = 21 + 20\sin(\theta(t))$$

$$y'(t) = 20\cos(\theta(t))\theta'(t).$$

As the wheel does one revolution every 1.5 minutes, so $\theta(t)$ increases by 2π when t increases by 1.5. Thus

$$\theta'(t) = 2\pi/1.5$$

$$33 = y(t_0) = 21 + 20\sin(\theta(t_0))$$

$$.6 = \sin(\theta(t_0))$$

$$-.8 = \cos(\theta(t_0))$$

$$y'(t_0) = -20(.8)(4\pi/3).$$

We use linear approximation. The rider is 33 meters high and rising at a rate of $y'(t_0) = -20(.8)(4\pi/3)$ meters per minute. Two seconds is $1/30$ minute.
Thus

$$y(t_0+1/30) \approx y(t_0)+(1/30)y'(t_0) = 33-20(.8)(4\pi/3)(1/30) \approx 31.77.$$

A lighthouse is located on a small island three kilometers away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving across the shoreline when it is twenty kilometers away from P? About how long will it take for the beam to get to a point on the shore 21 kilometers away from P?

We let $x(t)$ be the position of the light on the beach (with P at $x = 0$).

We let $\theta(t)$ be the angle between the line from the lighthouse to P and the line from the lighthouse to the light on the beach ($x(t)$).

$$\tan(\theta(t)) = \frac{x(t)}{3}$$

$$\sec^2(\theta(t))\theta'(t) = \frac{x'(t)}{3}.$$

Four revolutions per minute means that θ increases by 8π every minute and $\theta'(t) = 8\pi$.
As $x(t_0) = 20$

$$\cos(\theta(t_0)) = \frac{3}{\sqrt{20^2 + 3^2}}.$$

$$\sec^2(\theta(t_0)) = \frac{20^2 + 3^2}{9} = \frac{409}{9}.$$

$$\frac{409}{9}(8\pi) = \frac{x'(t_0)}{3}.$$

$$x'(t_0) = \frac{3272\pi}{3} \approx 3,426.43.$$

For the beam to move 1 kilometer x increases from 20 to 21.

$$21 = x(t_0 + \Delta t) \approx x(t_0) + x'(t_0)(\Delta t) \approx 20 + (3,426.43)(\Delta t).$$

$$1/(3426.43) \approx \Delta t.$$

So Δt is about .000291 minutes or .0175 seconds.