Logarithmic Differentiation

Sometime it is necessary (or easier) to take natural logs before taking the derivative. Then we proceed with implicit differentiation.
We use this when

- the function has a product or quotient of a large number of terms or
- there is an x in both the base and exponent.

Differentiate
$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5(2x+1)}.$$
Taking natural log of both sides we get
$$\ln(y) = \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) - \ln(2x+1).$$
$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\frac{3}{4}\ln(x)+\frac{1}{2}\ln(x^2+1)-5\ln(3x+2)-\ln(2x+1)\right).$$
$$\frac{1}{y}y' = \frac{3}{4}(1/x) + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{1}{3x+2}(3) - \frac{1}{2x+1}(2).$$
$$y' = \left(\frac{3}{4}(1/x)+\frac{1}{2}\frac{2x}{x^2+1}-5\frac{1}{3x+2}(3)-\frac{1}{2x+1}(2)\right)\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5(2x+1)}.$$

Power Rule

Prove the power rule by logarithmic differentiation.
If $f(x) = x^n$ then
$$\ln(f(x)) = \ln(x^n) = n\ln(x).$$
Taking derivatives we get
$$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(n\ln(x)).$$
$$\frac{1}{f(x)}f'(x) = \frac{n}{x}.$$
$$f'(x) = f(x)\frac{n}{x} = x^n\frac{n}{x} = nx^{n-1}.$$

Use logarithmic differentiation to find the derivative of $y = x^x$.
$$\ln(y) = x\ln(x).$$
$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x\ln(x)).$$
$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln(x)(1).$$
$$\frac{dy}{dx} = y(1+\ln(x)).$$
$$\frac{dy}{dx} = x^x(1+\ln(x)).$$

Derivatives of functions with exponents

If a and b are constants then

$$\frac{d}{dx}\left(a^b\right) = 0.$$

$$\frac{d}{dx}\left(a^{f(x)}\right) = \ln(a)a^{f(x)}f'(x).$$

$$\frac{d}{dx}\left(f(x)^b\right) = bf(x)^{b-1}f'(x).$$

$\frac{d}{dx}\left(f(x)^{g(x)}\right)$ use logarithmic differentiation

If $y = (2\sqrt{x})^{3x^2+2x}$ find y'.

$$\ln(y) = \ln((2\sqrt{x})^{3x^2+2x}).$$

$$\ln(y) = (3x^2+2x)\ln(2\sqrt{x}).$$

$$\frac{1}{y}\frac{dy}{dx} = (3x^2+2x)\frac{1}{2\sqrt{x}}2\frac{1}{2}x^{-1/2} + (6x+2)\ln(2\sqrt{x}).$$

$$\frac{dy}{dx} = y\left((3x^2+2x)\frac{1}{2\sqrt{x}}2\frac{1}{2}x^{-1/2} + (6x+2)\ln(2\sqrt{x})\right).$$

$$\frac{dy}{dx} = (2\sqrt{x})^{3x^2+2x}\left((3x^2+2x)\frac{1}{2\sqrt{x}}2\frac{1}{2}x^{-1/2} + (6x+2)\ln(2\sqrt{x}))\right).$$

If

$$y = \frac{(2x-5)\sqrt{x^2+3}(x^4+1)^{2.3}}{(x-2)(x^5-2)(3-7x)x^4}$$

find y'.

$$\ln(y) = \ln\left(\frac{(2x-5)\sqrt{x^2+3}(x^4+1)^{2.3}}{(x-2)(x^5-2)(3-7x)x^4}\right).$$

$$\ln(y) = \ln(2x-5) + \frac{1}{2}\ln(x^2+3) + 2.3\ln(x^4+1) - \ln(x-2)$$

$$-\ln(x^5-2) - \ln(3-7x) - 4\ln(x).$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\Bigg(\ln(2x-5)+\frac{1}{2}\ln(x^2+3)+2.3\ln(x^4+1)-\ln(x-2)$$

$$-\ln(x^5-2) - \ln(3-7x) - 4\ln(x)\Bigg).$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x-5}2 + \frac{1}{2}\frac{1}{x^2+3}2x + 2.3\frac{1}{x^4+1}4x^3 - \frac{1}{x-2}(1)$$

$$-\frac{1}{x^5-2}5x^4 - \frac{1}{3-7x}(-7) - 4\frac{1}{x}.$$

$$\frac{dy}{dx} = \left(\frac{(2x-5)\sqrt{x^2+3}(x^4+1)^{2.3}}{(x-2)(x^5-2)(3-7x)x^4}\right)\Bigg(\frac{1}{2x-5}2 + \frac{1}{2}\frac{1}{x^2+3}2x$$

$$+2.3\frac{1}{x^4+1}4x^3 - \frac{1}{x-2}(1) - \frac{1}{x^5-2}5x^4\frac{1}{3-7x}(-7) - 4\frac{1}{x}\Bigg).$$

$$y(x) = (5+2x^2)\sqrt{x^4+3}(x^4+1)^x.$$

Find $y'(x)$.

$$\ln(y(x)) = \ln\left((5+2x^2)\sqrt{x^4+3}(x^4+1)^x\right).$$

$$\ln(y(x)) = \ln(5+2x^2) + \frac{1}{2}\ln(x^4+3) + x\ln(x^4+1).$$

$$\frac{d}{dx}(\ln(y(x))) = \frac{d}{dx}\left(\ln(5+2x^2) + \frac{1}{2}\ln(x^4+3) + x\ln(x^4+1)\right).$$

$$\frac{1}{y(x)}\frac{dy}{dx} = \frac{1}{5+2x^2}(4x) + \frac{1}{2}\frac{1}{x^4+3}(4x^3) + x\frac{1}{x^4+1}(4x^3) + (1)\ln(x^4+1).$$

$$\frac{dy}{dx} = (5+2x^2)\sqrt{x^4+3}(x^4+1)^x$$

$$\left(\frac{1}{5+2x^2}(4x) + \frac{1}{2}\frac{1}{x^4+3} + x\frac{1}{x^4+1}(4x^3) + (1)\ln(x^4+1)\right).$$

Related Rates

An oil platform explodes and spills oil into the Gulf of Mexico. The oil spill grows in a circular fashion. If the radius of the spill is growing at a rate of 25 kilometers per day when the radius is 220 kilometers, how fast is the area of the spill growing?

Let $A(t)$ be the area of the spill.
Let $r(t)$ be the radius of the spill.

$$A(t) = \pi(r(t))^2.$$

$$A'(t) = 2\pi r(t)r'(t).$$

Let t_0 be the time when $r(t_0) = 220$. We are asked to find $A'(t_0)$. We have that

$$r(t_0) = 220 \quad \text{and} \quad r'(t_0) = 25.$$

Plugging into

$$A'(t) = 2\pi r(t)r'(t)$$

we get

$$A'(t) = 2\pi(220)(25) = 11000\pi.$$

Strategy

1. Draw a picture.
2. Choose notation.
3. Write an equation.
4. Differentiate using the chain rule.
5. Substitute into the equation and solve.

Related Rates

A baseball diamond is a square that is 90 feet on each side. A baseball player is running from first base to second base at a rate of 18 feet per second. How fast is the distance from home plate changing when the runner is midway from first base to second?

Let $d_1(t)$ be the distance of the runner to first base.
and let $d_{\text{home}}(t)$ be the distance of the runner to first base.
We are asked to find $d'_{\text{home}}(t_0)$ when $d_1(t_0) = 45$ and $d'_1(t_0) = 18$.
Then we have that

$$(d_1(t))^2 + 90^2 = (d_{\text{home}}(t))^2.$$

Now we implicitly differentiate

$$\frac{d}{dt}\left((d_1(t))^2 + 90^2\right) = \frac{d}{dt}\left(d_{\text{home}}(t))^2.\right)$$

$$2d_1(t)d'_1(t) = 2d_{\text{home}}(t)d'_{\text{home}}(t)$$

Plugging into the above equation we get

$$2(45)(18) = 2d_{\text{home}}(t_0)d'_{\text{home}}(t_0).$$

We first need to find $d_{\text{home}}(t_0)$.

$$(d_1(t_0))^2 + 90^2 = (d_{\text{home}}(t_0))^2.$$

$$(45)^2 + 90^2 = (d_{\text{home}}(t_0))^2.$$

$$45\sqrt{5} = d_{\text{home}}(t_0).$$

Plugging this value into our equation we get

$$2(45)(18) = 2(45\sqrt{5})d'_{\text{home}}(t_0).$$

$$d'_{\text{home}}(t_0) = \frac{2(45)(18)}{(2)45\sqrt{5}} = \frac{18}{\sqrt{5}}.$$

Related Rates

A baseball diamond is a square that is 90 feet on each side. One player runs from first to second and another runs from second to third. The one running from first is running at a rate of 18 feet per second while the one running from second is running at a rate of 20 feet per second. How fast is the distance between the two runners changing when the first is 30 feet from first base and the second is 40 feet from second?

Let $x(t)$ be the distance from the first runner and second base.
Let $y(t)$ be the distance from the second runner and second base.
Let $d(t)$ be the distance between the two runners.
Then we have the equation

$$d(t)^2 = x(t)^2 + y(t)^2.$$

$$2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t).$$

Let t_0 be the time when the first is 30 feet from first base and the second is 40 feet from second. We are asked to find $d'(t_0)$
We have that

$$x(t_0) = 60 \quad \text{and} \quad y(t_0) = 40.$$

Thus

$$d(t_0) = \sqrt{x(t_0)^2 + y(t_0)^2} = \sqrt{(60)^2 + (40)^2} = 20\sqrt{13}.$$

We are also given that $x'(t_0) = -18$ and $y'(t_0) = 20$.

Plugging all of this into

$$2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

we get

$$2(20\sqrt{13})d'(t_0) = 2(60)(-18) + 2(40)(20).$$

Solving we get

$$d'(t_0) = \frac{2(60)(-18) + 2(40)(20)}{2(20\sqrt{13})} =$$

Related Rates

Air is leaking out of a balloon. The balloon is always spherical and its volume is decreasing at a rate of 100 cc/min. When the balloon has a volume of 2000 cc how quickly is the radius decreasing?

$$V(t) = \frac{4}{3}\pi r^3(t)$$

$$V'(t) = 4\pi r^2(t) r'(t)$$

$V(t_0) = 2000$ so $r(t_0) = \sqrt[3]{\frac{1500}{\pi}}$.

$$-100 = 4\pi\left(\sqrt[3]{\frac{1500}{\pi}}\right)^2 2r'(t)$$

$$r'(t) = \frac{-300(\pi)^{2/3}}{4(1500)^{2/3}(2)} \approx$$

Related Rates

Two variable resistors are connected in parallel. If the two resistors have resistance R_1 and R_2 then the effective resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 and R_2 are increasing at rates of $.3\Omega/\text{sec}$ and of $.2\Omega/\text{sec}$ respectively then how fast is R changing when $R_1 = 50$ and $R_2 = 80$?

$$\frac{1}{R(t)} = \frac{1}{R_1(t)} + \frac{1}{R_2(t)}.$$

$$\frac{-R'(t)}{(R(t))^2} = \frac{-R_1'(t)}{(R_1(t))^2} + \frac{-R_2'(t)}{(R_2(t))^2}.$$

If t_0 is the time that we are interested in then we have $R_1'(t_0) = .3$ and $R_2'(t) = .2$ $R_1(t_0) = 50$ and $R_2(t_0) = 80$.

$$\frac{-R'(t_0)}{(R(t_0))^2} = \frac{-.3}{50^2} + \frac{-.2}{80^2}.$$

We need to find

$$R(t_0) = \frac{1}{R_1(t_0)} + \frac{1}{R_2(t_0)} = \frac{1}{50} + \frac{1}{80} = \frac{13}{400}.$$

$$\frac{-R'(t_0)}{(13/400)^2} = \frac{-.3}{50^2} + \frac{-.2}{80^2}.$$

$$R'(t_0) = -\frac{400^2}{13^2}\left(\frac{-.3}{50^2} + \frac{-.2}{80^2}\right) \approx .$$

Strategy

1. Draw a picture
2. Choose notation
3. Write an equation
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Related Rates

A Ferris wheel with a radius of 20 meters is rotating at a rate of one revolution every 1.5 minutes. The lowest a seat goes is 1 meter above the ground. How fast is a rider rising when his seat is 33 meters above the ground?

$$y(t) = 21 + 20\sin(\theta(t))$$
$$y'(t) = 20\cos(\theta(t))\theta'(t)$$

$$\theta'(t) = 2\pi/1.5$$
$$33 = y(t_0) = 21 + 20\sin(\theta(t_0))$$

$$.6 = \sin(\theta(t_0))$$
$$.8 = \cos(\theta(t_0))$$
$$y'(t) = 20(.8)(4\pi/3)$$

Related Rates

A lighthouse is located on a small island three kilometers away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving across the shoreline when it is one kilometer away from P.

$$\tan(\theta(t)) = \frac{x(t)}{3}$$
$$\sec^2(\theta(t))\theta'(t) = \frac{x'(t)}{3}.$$

Four revolutions per minute means that θ increases by 8π every minute and $\theta'(t) = 8\pi$.
As $x(t_0) = 20$

$$\cos(\theta(t_0)) = \frac{3}{\sqrt{20^2 + 3^2}}.$$
$$\sec^2(\theta(t_0)) = \frac{20^2 + 3^2}{9}.$$
$$\sec^2(\theta(t))\theta'(t) = \frac{x'(t)}{3}.$$
$$\frac{409}{9}(8\pi) = \frac{x'(t_0)}{3}.$$
$$\frac{3272\pi}{3} = x'(t_0).$$