Haim's Notes About *Introduction to Microlocal Analysis* by Peter Hintz

Table of Contents

1 Notations and Conventions

Convention 1.1: I use the Einstein summation convention.

Convention 1.2: Unless specified otherwise, "smooth" means C^∞.

2 Chapter 5

2.1 Definition of Distributions on Manifolds (Definition 5.15)

Hintz defines distributions on manifolds as continuous linear functionals over $C_c^\infty(M; E^* \otimes \Omega M)$ or $C^\infty(M; E^* \otimes \Omega M)$, but I can't find where the author defines the mentioned notion of "continuity." In this note, I'd like to define and discuss this continuity. Furthermore, Lars Hörmander on page 144 of Volume 1 in his work *The Analysis of Linear Partial Differential Operators* (c.f. [1]) describes an alternative definition of distributions. As a second objective in this note, I'd like to connect the two definitions provided by Hörmander and Hintz. The first part of the following is essentially me generalizing the exposition of distributions done in Friedlander and Joshi's book *Introduction to the Theory of Distributions* (c.f. [2]) to the context of manifolds.

* In this section all functions are complex valued and vector bundles are over the field of complex numbers (i.e. I don't restrict only to real vector bundles since I don't see a reason to).

In this section, let M be a smooth manifold without boundary and let (π, F, M) be a smooth vector bundle over M of rank k. We start with describing the topology of $C_c^\infty(M; F)$. We won't actually define a topology on $C_c^\infty(M; F)$, but simply declare what we mean by the words "continuous" and "convergent" in this context.

Definition 2.1: We say that a linear functional of the form $u : C_c^\infty(M;F) \to \mathbb{C}$ is continuous if for any coordinate chart $(U,(x^i))$ ($U \subseteq M$ denotes the domain of the chart), for any frame (b_j) of F over U, and any compact set $K \subseteq U$ there exists a real number $C > 0$ and a nonnegative integer $N \geq 0$ such that

$$|\langle u,\phi\rangle| \leq C\sum_{j=1}^{k}\sum_{|\alpha|\leq N}\sup\left|\partial^\alpha\phi^j\right| \qquad \forall\phi : \operatorname{supp}\phi \subseteq K$$

Where the ϕ^j denote the components of ϕ with respect to (b_j): $\phi = \phi^j b_j$.

//

Definition 2.2: We say that a sequence $\{\phi_r\}$ in $C_c^\infty(M;F)$ converges to $\phi \in C_c^\infty(M;F)$ as $r \to \infty$ if there exists a compact set $K \subseteq M$ that contains all of the $\operatorname{supp}\phi_r$ and if for any coordinate chart $(U,(x^i))$, any frame (b_j) of F over U, any multi-index α, and any compact set $Q \subseteq U$, we have $\partial^\alpha\phi_r^j$ converges to $\partial^\alpha\phi^j$ uniformly over Q as $r \to \infty$.

//

Theorem 2.3 *A linear functional $u : C_c^\infty(M;F) \to \mathbb{C}$ is continuous if and only if it is sequentially continuous in the sense of Definition 2.2 above.*

Proof: First suppose that u is continuous. Take any sequence $\{\phi_r\}$ converging to ϕ in $C_c^\infty(M;F)$. Let $K \subseteq M$ be a compact subset containing all of the $\operatorname{supp}\phi_r$. Cover K by coordinate charts $\{(U_s,(x_s^i)) : s = 1,\dots,m\}$ with frames (b_j^s) of F over U_s for $s = 1,\dots,m$. Let $\mathcal{U} = \bigcup_{s=1}^m U_s$ and let $\{\rho_s : \mathcal{U} \to \mathbb{R} : s = 1,\dots,m\}$ be a partition of unity subordinate to the cover $\{U_s\}$. By linearity we have that

$$\langle u,\phi_r\rangle = \sum_{s=1}^{m}\langle u,\rho_s\phi_r\rangle$$

and similarly with ϕ_r replaced by ϕ. From this we see that it will be suffice to show that $\langle u,\rho_s\phi_r\rangle \to \langle u,\rho_s\phi\rangle$ as $r \to \infty$ for every $s = 1,\dots,m$. Fix any $s = 1,\dots,m$. By the definition of u's continuity, there exist $C > 0$ and $N > 0$ such that in the coordinates (x_s^i) and frame (b_j^s)

$$|\langle u,\psi\rangle| \leq C\sum_{j=1}^{k}\sum_{|\alpha|\leq N}\sup\left|\partial^\alpha\psi^j\right| \qquad \forall\psi : \operatorname{supp}\psi \subseteq \operatorname{supp}\rho_s.$$

Hence by the product rule and the fact that ρ_s and all of its partials are bounded, we get that

$$|\langle u,\rho_s(\phi_r-\phi)\rangle| \leq C\sum_{j=1}^{k}\sum_{|\alpha|\leq N}\sup\left|\partial^\alpha\left[\rho_s\left(\phi_r^j-\phi^j\right)\right]\right| \leq C'\sum_{j=1}^{k}\sum_{|\alpha|\leq N}\sup_{\operatorname{supp}\rho_s}\left|\partial^\alpha\left[\phi_r^j-\phi^j\right]\right|$$

for some constant $C' > 0$ independent of r. Since by assumption each $\partial^\beta \phi_r^j$ converges uniformly to $\partial^\beta \phi^j$ over $\operatorname{supp} \rho_s$, this shows that indeed $\langle u, \rho_s \phi_r \rangle \to \langle u, \rho_s \phi \rangle$ as $r \to \infty$.

Now let's prove the other direction: suppose that u is sequentially continuous in the sense of Definition 2.2 above. We will prove that u is continuous by contradiction: suppose not! Then there exists a coordinate chart $(U, (x^i))$, a frame (b_j) of F over U, and a compact set $K \subseteq U$ such that for any integer $C > 0$ and any $N > 0$ there exists a nonzero $\psi_{C,N} \in C_c^\infty(M; F)$ such that $\operatorname{supp} \psi_{C,N} \subseteq K$ and

$$(2.4) \qquad |\langle u, \psi_{C,N} \rangle| > C \sum_{j=1}^{k} \sum_{|\alpha| \le N} \sup |\partial^\alpha \psi_{C,N}^j|.$$

Consider the sequence $\{\phi_r\}$ for $r = 1,2,3, \ldots$ given by

$$\phi_r = \frac{\psi_{r,r}}{r \sum_{j=1}^{k} \sum_{|\alpha| \le r} \sup |\partial^\alpha \psi_{r,r}^j|}.$$

All of the $\operatorname{supp} \phi_r$ are contained in K and it's clear from the above expression that for every multi-index β we have that $\partial^\beta \phi_r \to 0$ uniformly. It's not hard to see then that $\phi_r \to 0$ in $C_c^\infty(M; F)$ and hence by u's continuity we have that $\langle u, \phi_r \rangle \to 0$. But that's a contradiction since by (2.4) above we have that each $|\langle u, \phi_r \rangle| > 1$. Hence indeed u must be continuous.

■

Definition 2.5: Suppose that $u : C_c^\infty(M; F) \to \mathbb{C}$ is a continuous linear functional. We define the **distributional support** of u, denoted by $\operatorname{supp} u$, to be the complement of the following set

$$\{x \in M : \exists \text{ open neighborhood } U \text{ of } x \text{ such that } \langle u, \phi \rangle = 0 \ \ \forall \phi \in C_c^\infty(M; F) : \operatorname{supp} \phi \subseteq U\}.$$

//

Warning: The distributional support of u is not the support of u in the topology of $C_c^\infty(M; F)$. Furthermore, the term "distributional support" is not standard but will simply be used in this section. When talking about distributions, people call the "distributional support" simply the "support" of u.

Definition 2.6: Suppose that $\{u_r : C_c^\infty(M; F) \to \mathbb{C}$ where $r = 1,2,3, \ldots\}$ is a sequence of continuous linear functionals and that $u : C_c^\infty(M; F) \to \mathbb{C}$ is a continuous linear functional. We say that the sequence $\{u_r\}$ converges to u if for all $\phi \in C_c^\infty(M; F)$, $\langle u_r, \phi \rangle \to \langle u, \phi \rangle$ as $r \to \infty$.

//

Definition 2.7: Suppose that $u : C_c^\infty(M; F) \to \mathbb{C}$ is a continuous linear functional and that $\phi \in C^\infty(M)$. We define the left-multiplication ϕu to be the continuous linear functional $\phi u : C_c^\infty(M; F) \to \mathbb{C}$ given by

$$\langle \phi u, \psi \rangle = \langle u, \phi \psi \rangle \qquad \forall \psi \in C_c^\infty(M; F).$$

We leave it to the reader to check the claim that ϕu is indeed continuous.

//

Next we transition to discussing what it means for a linear functional over $C^\infty(M;F)$ to be continuous. For $C^\infty(M;F)$ it's actually not hard to define the topology and so, unlike above, we can define a genuine notion of continuity and convergence in this context.

Definition 2.8: For any coordinate chart $(U,(x^i))$, for any frame (b_j) of F over U, for any compact set $K \subseteq U$, for any $l = 1, \dots, k$, and any multi-index α, we let $p_{(U,(x^i)),(b_j),K,l,\alpha} : C^\infty(M;F) \to [0,\infty)$ denote the seminorm given by

$$p_{(U,(x^i)),(b_j),K,l,\alpha}(\phi) = \sup_{x\in K} |\partial^\alpha \phi^l|.$$

We define the topology on $C^\infty(M;F)$ to be the one generated by such seminorms.

//

Theorem 2.9: *With the topology above, $C^\infty(M;F)$ is a Fréchet space.*

Proof: It will be sufficient to show that

1. We can pick out a countable family of the seminorms in Definition 2.8 above that determines the same topology,
2. Any Cauchy sequence in $C^\infty(M;F)$ converges (i.e. $C^\infty(M;F)$ is complete).

Let's start with showing (1). Let $\{(U_r,(x_r^i)) : r = 1,2,3,\dots\}$ be a countable cover of M and let (b_j^r) be a frame of F over each U_r. For each r, let $\{K_{r,s} : s = 1,2,3,\dots\}$ be a compact exhaustion of U_r and consider the countable family of seminorms $\{p_{r,s,v,\beta} = p_{(U_r,(x_r^i)),(b_j^r),K_{r,s},v,\beta}\}$. Now, take any coordinate chart $(U,(x^i))$, any frame (b_j) of F over U, any compact set $K \subseteq U$, any $l = 1, \dots, k$, any multi-index α, and consider their associated seminorm $p = p_{(U,(x^i)),(b_j),K,l,\alpha}$. It's not hard to see that if we can show that p can be bounded by a finite sum of the $p_{r,s,\beta}$'s, we'll be done. Let $\{K_{r_t,s_t} : t = 1, \dots, m\}$ be a finite subset such that the interiors of the K_{r_r,s_t} cover K. Then for any $\phi \in C^\infty(M;F)$ we have that

$$p(\phi) = \sup_{x\in K} |\partial_x^\alpha \phi^l| \le \sum_{t=1}^m \sup_{x\in K\cap K_{r_t,s_t}} |\partial_x^\alpha \phi^l| \le \sum_{t=1}^m \sup_{x\in K_{r_t,s_t}} |\partial_x^\alpha \phi^l|.$$

Now, over K_{r_t,s_t} the partial ∂_x^α can be expressed as a big linear combination of partials of the form $\partial_{x_r}^\beta$ for $|\beta| \le |\alpha|$ with coefficients in terms of $\partial x^\mu / \partial x_r^\nu$ and their partials over K_{r_t,s_t}. Observe that such coefficients are bounded over K_{r_t,s_t}. Furthermore, over K_{r_t,s_t} the "l^{th}" component of a section of F with respect to (b_j) (e.g. "ϕ^l") can be expressed as a linear combination of the components of the same section with respect to $(b_j^{r_t})$ with coefficients in terms of the transition matrix from $(b_j^{r_t})$ to (b_j) and their partials. Such coefficients are also bounded over K_{r_t,s_t}. It's not hard to see that this implies that there exists some $C > 0$ independent of ϕ such that (here ϕ^v in each term is the component of ϕ with respect to $(b_j^{r_t})$)

$$p(\phi) \leq C \sum_{t=1}^{m} \sum_{v=1}^{k} \sum_{|\beta|\leq|\alpha|} \sup_{x\in K_{r_t,s_t}} \left|\partial_{x_r}^{\beta} \phi^v\right| = C \sum_{t=1}^{m} \sum_{v=1}^{k} \sum_{|\beta|\leq|\alpha|} p_{r_t,s_t,v,\beta}(\phi).$$

Now let's prove (2). Suppose that $\{\phi_\mu\}$ is a sequence such that each $p_{r,s,v,\beta}(\phi_\mu - \phi_\nu) \to 0$ as $\mu, \nu \to \infty$. We have to show that $\{\phi_j\}$ converges to some ϕ in $C^\infty(M; F)$. By assumption then we immediately have that each

$$\sup_{x\in K_{r,s}} \left|\partial_{x_r}^{\beta}[\phi_\mu^v - \phi_\nu^v]\right| \to 0 \qquad \text{as} \quad \mu, \nu \to \infty.$$

In other words, $\{\phi_\mu^v\}$ is a sequence uniformly convergent in all partials over a compact exhaustion of U_r. It's well known that this implies that there exists a function $\phi_r^v \in C^\infty(U_r)$ such that $\sup |\partial_{x_r}^{\beta}\phi_\mu^v - \partial_{x_r}^{\beta}\phi^v| \to 0$ as $\mu \to \infty$. We leave it to the reader to show that for any r, r' such that $U_r \cap U_{r'} \neq \emptyset$, $\phi_r^j b_j^r = \phi_{r'}^j b_j^{r'}$ (no implicit summation in r or r'), which in fact quickly follows from the continuity of the transition charts for F. Hence we can patch up to get a well-defined smooth section $\phi \in C^\infty(M; F)$ such that each

$$p_{r,s,v,\beta}(\phi_\mu - \phi) = \sup_{x\in K_{r,s}} \left|\partial_{x_r}^{\beta}[\phi_\mu^v - \phi^v]\right| \to 0 \qquad \text{as} \quad \mu \to \infty.$$

In other words, $\{\phi_\mu\}$ converges to ϕ in $C^\infty(M; F)$ and hence we're done.

■

Having defined the topology of $C^\infty(M; F)$, it's clear what we mean by a linear functional $u : C^\infty(M; F) \to \mathbb{C}$ to be continuous. Since $C^\infty(M; F)$ is a Fréchet space, we automatically get that such a u is continuous if and only if its sequentially continuous.

Definition 2.10: We impose the weak* topology on the set of continuous linear functional of the form $u : C^\infty(M; F) \to \mathbb{C}$.

//

We define left multiplication of smooth functions on continuous linear functionals over $C^\infty(M; F)$ analogously to Definition 2.7 above:

Definition 2.11: Suppose that $u : C^\infty(M; F) \to \mathbb{C}$ is a continuous linear functional and that $\phi \in C^\infty(M)$. We define the left-multiplication ϕu to be the continuous linear functional $\phi u : C^\infty(M; F) \to \mathbb{C}$ given by

$$\langle \phi u, \psi \rangle = \langle u, \phi\psi \rangle \qquad \forall \psi \in C^\infty(M; F).$$

We leave it to the reader to check the claim that ϕu is indeed continuous.

//

It turns out that we can think of the set of continuous linear functionals of compact distributional support as the set of linear functionals over $C^\infty(M;F)$. The following theorem tells us how we make such an identification.

Theorem 2.12: *The following are true:*

1. *If $u : C_c^\infty(M;F) \to \mathbb{C}$ is a continuous linear functional of compact distributional support, then there exists a unique continuous linear functional $v : C^\infty(M;F) \to \mathbb{C}$ that extends u.*
2. *If $v : C^\infty(M;F) \to \mathbb{C}$ is a continuous linear functional, then v restricts to a continuous linear functional u over $C_c^\infty(M;F)$ of compact distributional support.*

Proof: I will write "support" instead of "distributional support" in this proof for brevity. Item (2) follows if one shows that u is sequentially continuous, which will follow if one shows that a sequence converging in $C_c^\infty(M;F)$ implies that it converges in $C^\infty(M;F)$. It's not hard to see however that the latter is true simply by looking at Definition 2.2 and Definition 2.8 above.

Hence let's just show (1). Suppose that $u : C_c^\infty(M;F) \to \mathbb{C}$ is a continuous linear functional with compact support. Let $\rho : C_c^\infty(M)$ be a function that is identically one in a neighborhood of the support of u. For any smooth section $\phi \in C_c^\infty(M;F)$ we have that

$$\langle u, \phi\rangle = \langle u, \rho\phi\rangle + \langle u, (1-\rho)\phi\rangle.$$

Since $\operatorname{supp}(1-\rho)\phi \subseteq M \setminus \operatorname{supp} u$, we have by the definition of the support of u that the second term on the right-hand side is zero. Hence if we define $v : C^\infty(M;F) \to \mathbb{C}$ to be the linear functional $\langle v, \psi\rangle = \langle u, \rho\psi\rangle$ for all $\psi \in C^\infty(M;F)$, then v will extend u.

Let's show that v is continuous. Let $\{(U_r, (x_r^i)) : r = 1, \dots, m\}$ be a finite collection of coordinates of M that cover $\operatorname{supp}\rho$ and let $\{(b_j^r) : r = 1, \dots, m\}$ be a finite collection of frames for F over the U_r respectively. Let $\mathcal{U} = \bigcup_{r=1}^m U_r$ and let $\{\theta_r : U \to \mathbb{R} \text{ for } r = 1, \dots, m\}$ be a partition of unity subordinate to $\{U_r\}$. Without loss of generality, we can suppose that we chose each U_r to have compact closure and so each $\operatorname{supp}\theta_r$ is compact. Now, we have that for any $\phi \in C^\infty(M;F)$

$$|\langle v, \phi\rangle| = |\langle u, \rho\phi\rangle| \le \sum_{r=1}^m |\langle u, \rho\theta_r\phi\rangle|.$$

Observe that for any $\phi \in C^\infty(M;F)$, the support of each $\rho\theta_r\phi$ is contained in the compact set $\operatorname{supp}\theta_r$. Hence by the continuity of u we see that there exist $C, N > 0$ such that for any $\phi \in C^\infty(M;F)$ the above quantity is bounded by

$$C\sum_{r=1}^m \sum_{j=1}^k \sum_{|\alpha|\le N} \sup\left|\partial_{x_r}^\alpha\left[\rho\theta_r\phi^j\right]\right|$$

where for every r, $\rho\theta_r\phi^j$ is the component of $\rho\theta_r\phi$ with respect to (b_j^r). By applying the product rule to $\partial_{x_r}^{\alpha}\left[\rho\theta_r\phi^j\right]$ and using that $\rho\theta_r$ and its partials are bounded, we arrive that for some other $\tilde{C} > 0$

$$|\langle v, \phi \rangle| \leq \tilde{C} \sum_{r=1}^{m} \sum_{j=1}^{k} \sum_{|\alpha| \leq N} \sup_{x \in \operatorname{supp} \theta_r} \left|\partial_{x_r}^{\alpha} \phi^j\right| \qquad \forall \phi \in C^{\infty}(M; F).$$

Hence indeed v is continuous.

The only thing left to show is that v is the unique such continuous extension. To see why, suppose that $\tilde{v}$ was another. Then $v - \tilde{v}$ is also continuous. Hence by definition there exist a finite collection $\{(U_r, (x_r^i)) : r = 1, \dots, m\}$ of coordinates of M, a finite collection $\{(b_j^r) : r = 1, \dots, m\}$ of frames for F over the U_r respectively, a finite collection $\{K_r \subseteq U_r : r = 1, \dots, m\}$ of compact sets, and constants $C, N > 0$ such that

$$|\langle v - \tilde{v}, \phi \rangle| \leq C \sum_{r=1}^{m} \sum_{j=1}^{k} \sum_{|\alpha| \leq N} \sup_{x \in K_r} \left|\partial_{x_r}^{\alpha} \phi^j\right| \qquad \forall \phi \in C^{\infty}(M; F),$$

where for every r, ϕ^j in $\sup\left|\partial_{x_r}^{\alpha}\phi^j\right|$ is the component of ϕ with respect to (b_j^r). From this equation we see that if $\phi \in C^{\infty}(M; F)$ is such that $\operatorname{supp} \phi$ is disjoint from all of the K_r, then $v - \tilde{v}$ applied to it will be zero. Hence, if take a compactly supported $\rho \in C_c^{\infty}(M)$ that is identically one on all of the K_r, we see that

$$\langle v - \tilde{v}, \phi \rangle = \langle v - \tilde{v}, \rho\phi \rangle = \langle u - u, \rho\phi \rangle = 0.$$

Hence indeed $v = \tilde{v}$.

■

We note that if $u : C_c^{\infty}(M; F) \to \mathbb{C}$ and $v : C^{\infty}(M; F) \to \mathbb{C}$ are continuous linear functionals where v is the unique extension of u as described in Theorem 2.12 above and $\phi \in C^{\infty}(M)$ is a smooth function, then it's not hard to see that ϕv is the unique continuous extension of ϕu.

Furthermore, we remark that because of the Theorem 2.12 above, if we have a continuous linear functional $u : C_c^{\infty}(M; F) \to \mathbb{C}$ of compact distributional support then it's standard to use the same letter u to denote its unique continuous extension to $C^{\infty}(M; F)$.

Theorem 2.13: *It holds that*

1. *$C_c^{\infty}(M; F)$ is dense in $C^{\infty}(M; F)$,*
2. *Linear functionals over $C_c^{\infty}(M; F)$ of compact distributional support are dense in the space of continuous linear functionals over $C_c^{\infty}(M; F)$.*

Proof: Let $\{K_r : r = 1,2,3, \dots\}$ be a compact exhaustion of M and let $\{\rho_r \in C_c^{\infty}(M) : r = 1,2,3, \dots\}$ be such that each $\rho_r \equiv 1$ on K_r. To prove (1), take any $\phi \in C^{\infty}(M; F)$ and consider the sequence $\{\rho_r\phi \in C_c^\infty(M;F) : r = 1,2,3,\dots\}$. We claim that $\rho_r\phi \to \phi$ in $C^\infty(M;F)$. To prove this, it's sufficient to take any coordinate chart $(U,(x^i))$, any frame (b_j) of F over U, any compact set $K \subseteq U$, any $l = 1,\dots,k$, any multi-index α, and show that

$$p_{(U,(x^i)),(b_j),K,l,\alpha}(\rho_r\phi) = \sup_{x\in K}|\partial^\alpha[\rho_r\phi^l - \phi^l]| \to 0 \quad \text{as} \quad r \to \infty.$$

But since the support of ρ_r will eventually cover all of K as $r \to \infty$, this obviously holds.

To prove (2), take any continuous linear functional $u : C_c^\infty(M;F) \to \mathbb{C}$. Consider the sequence $\{\rho_r u \in C_c^\infty(M;F) : r = 1,2,3,\dots\}$. It's not hard to see that each $\rho_r u$ is of compact distributional support. Now, take any $\phi \in C_c^\infty(M;F)$. Eventually the support of ρ_r will cover $\operatorname{supp}\phi$ as $r \to \infty$, and hence eventually

$$\langle \rho_r u, \phi\rangle = \langle u, \rho_r\phi\rangle = \langle u, \phi\rangle.$$

In particular, we get that $\langle \rho_r u, \phi\rangle \to \langle u, \phi\rangle$ as $r \to \infty$ for all $\phi \in C_c^\infty(M;F)$. This implies that $\rho_r u \to u$ and hence the theorem is proved.

■

Having discussed continuity of linear functionals over $C_c^\infty(M;F)$ and $C^\infty(M;F)$, the notations $\mathcal{D}(M)$, $\mathcal{E}(M)$, $\mathcal{D}(M;E)$, and $\mathcal{E}(M;E)$ in Hintz' Definition 5.14 make sense.

Finally, let's discuss Hörmander's definition of distributions and relate it to Hintz' Definition.

Definition 2.14: (Hörmander's alternative definition of $\mathcal{D}'(M)$) Suppose that to every smooth chart (U,φ) of M (i.e. $\varphi : U \to \varphi[U] \subseteq \mathbb{R}^{\dim M}$ is the chart) we assign a "classical" distribution $u_{(U,\varphi)} : C_c^\infty(\varphi[U]) \to \mathbb{C}$ as defined in §1.3 in [2]. Suppose furthermore that we assigned them so that for any two charts (U,φ) and (U',φ') such that $U \cap U' \neq \emptyset$ (I omit writing appropriate domain restrictions in the following equation),

$$u_{(U,\varphi)} = (\varphi' \circ \varphi)^* u_{(U',\varphi')}$$

where the pullback is defined in §7.1 in [2]. Then we call this set $\{u_{(U,\varphi)} : (U,\varphi) \text{ smooth chart of } M\}$ a distribution in $\mathcal{D}'(M)$.

//

There is a canonical one-to-one correspondence between Hörmander's and Hintz's definitions of $\mathcal{D}'(M)$, which we state in the next theorem. To get ready, let $\mathcal{D}'_{\text{Hörm}}(M)$ and $\mathcal{D}'_{\text{Hintz}}(M)$ denote Hörmander's and Hintz's definitions of distributions respectively. We proceed to define maps $\mathcal{F} : \mathcal{D}'_{\text{Hörm}}(M) \to \mathcal{D}'_{\text{Hintz}}(M)$ and $\mathcal{G} : \mathcal{D}'_{\text{Hintz}}(M) \to \mathcal{D}'_{\text{Hörm}}(M)$ that we will show are inverses of each other and hence provide the desired canonical one-to-one identification between $\mathcal{D}'_{\text{Hörm}}(M)$ and $\mathcal{D}'_{\text{Hintz}}(M)$.

For any $\{u_{(U,\varphi)}\} \in \mathcal{D}'_{\text{Hörm}}(M)$ as in Definition 2.14 above, let $\mathcal{F}(u) : C_c^\infty(M;\Omega M) \to \mathbb{C}$ be the following distribution. Take any $\phi \in C_c^\infty(M;\Omega M)$. Cover $\operatorname{supp}\phi$ by charts $\{(U_s,\varphi_s = (x_s^i)) :$

$s = 1, \dots, m\}$ of M. Let $\mathcal{U} = \bigcup_{s=1}^{m} U_s$ and let $\{\rho_s : \mathcal{U} \to \mathbb{R} : s = 1, \dots, m\}$ be a partition of unity subordinate to the cover $\{U_s\}$. For each $s = 1, \dots, m$, let ϕ_s denote the component of ϕ with respect to $|dx_s^1 \wedge \dots \wedge dx_s^{\dim(M)}|$. Then we set

$$(2.15) \qquad \langle \mathcal{F}(u), \phi \rangle = \sum_{s=1}^{m} \langle u_{(U_s,(x_s^i))}, \rho_s \phi_s \circ \varphi_s \rangle.$$

We need to show that this is well defined. Suppose that $\{(U_r', \varphi_r' = (x_r'^i)) : r = 1, \dots, m'\}$ and $\{\rho_r : \mathcal{U} \to \mathbb{R} : r = 1, \dots, m'\}$ are similar objects as above. We need to show that

$$(2.16) \qquad \sum_{s=1}^{m} \langle u_{(U_s,(x_s^i))}, \rho_s \phi_s \circ \varphi_s \rangle = \sum_{r=1}^{m'} \langle u_{(U_r,(x_r'^i))}, \rho_r' \phi_r' \circ \varphi_r' \rangle$$

where ϕ_r' is the component of ϕ with respect to $|dx_r'^1 \wedge \dots \wedge dx_r'^{\dim(M)}|$. The right-hand side is equal to

$$\sum_{r=1}^{m'} \sum_{s=1}^{m} \langle u_{(U_r,(x_r'^i))}, \rho_s \rho_r' \phi_r' \circ \varphi_r' \rangle$$

$$= \sum_{r=1}^{m'} \sum_{s=1}^{m} \langle (\varphi_r' \circ \varphi_s^{-1})^* u_{(U_r,(x_r'^i))}, |\det D(\varphi_r' \circ \varphi_s^{-1})|^{-1} \rho_s \rho_r' \phi_r' \circ \varphi_s \rangle.$$

Since $u_{(U_s,(x_s^i))} = (\varphi_r' \circ \varphi_s^{-1})^* u_{(U_r,(x_r'^i))}$ and $\phi_s = |\det D(\varphi_r' \circ \varphi_s^{-1})|^{-1} \phi_r'$, this is equal to

$$\sum_{r=1}^{m'} \sum_{s=1}^{m} \langle u_{(U_s,(x_s^i))}, \rho_s \rho_r' \phi_s \circ \varphi_s \rangle = \sum_{s=1}^{m} \langle u_{(U_s,(x_s^i))}, \rho_s \phi_s \circ \varphi_s \rangle$$

and hence (2.16) is established.

Next let's show that $\mathcal{F}(u)$ is continuous. Take any coordinate chart $(U, (x^i))$, any frame (b_j) of F over U, and any compact set $K \subseteq U$. Since $u_{(U,(x^i))}$ is a distribution as defined in [2], there exist $C, N > 0$ such that

$$\left| \langle u_{(U,(x^i))}, \psi \rangle \right| \le C \sum_{|\alpha| \le N} \sup |\partial^\alpha \psi| \quad \forall \psi \in C_c^\infty(M) : \operatorname{supp} \psi \subseteq K.$$

Then for any $\phi \in C_c^\infty(M; \Omega M)$ such that $\operatorname{supp} \phi \subseteq K$, if we let ϕ_1 denote the component of ϕ with respect to $|dx^1 \wedge \dots \wedge dx^{\dim(M)}|$, then it's not hard to see that (2.15) implies that

$$|\langle u, \phi \rangle| = \left| \langle u_{(U,(x^i))}, \phi_1 \rangle \right| \le C \sum_{|\alpha| \le N} \sup |\partial^\alpha \phi_1|.$$

Hence indeed $\mathcal{F}(u)$ is continuous and thus an element of $\mathcal{D}_{\mathrm{Hintz}}'(M)$.

Now take any $u \in \mathcal{D}'_{\text{Hintz}}(M)$ and let $\mathcal{G}(u)$ be the following element of $\mathcal{D}'_{\text{Hörm}}(M)$. For any chart $(U, (x^i) = \varphi)$ and any $\psi \in C_c^\infty(\varphi[U])$ set

$$\langle \mathcal{G}(u)_{(U,(x^i))}, \psi \rangle = \langle u, \psi \circ \varphi |dx^1 \wedge \ldots \wedge dx^{\dim M}| \rangle.$$

Let's check that this is indeed a distribution as defined in [2]. Choose any compact set $\widehat{K} \subseteq \varphi[U]$. Since u is continuous, there exist $C, N > 0$ satisfying that for any $\phi \in C_c^\infty(M; \Omega M)$ such that $\operatorname{supp} \phi \subseteq \varphi^{-1}[\widehat{K}]$

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi_1|$$

where ϕ_1 denotes the component of ϕ with respect to $|dx^1 \wedge \ldots \wedge dx^{\dim(M)}|$. Hence for any $\psi \in C_c^\infty(\varphi[U])$ such that $\operatorname{supp} \psi \subseteq \widehat{K}$,

$$|\langle \mathcal{G}(u)_{(U,(x^i))}, \psi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \psi|,$$

and so indeed $\mathcal{G}(u)$ is a distribution as defined in [2].

Theorem 2.17: *The maps $\mathcal{F} : \mathcal{D}'_{\text{Hörm}}(M) \to \mathcal{D}'_{\text{Hintz}}(M)$ and $\mathcal{G} : \mathcal{D}'_{\text{Hintz}}(M) \to \mathcal{D}'_{\text{Hörm}}(M)$ defined above are inverses of each other.*

Proof: We leave it to the reader as an exercise. It's not hard: the above discussion essentially has all of the details.

2.2 Proof of Schwartz Kernel Theorem

Hintz states the Schwartz kernel theorem on manifolds in Theorem 5.16, however he doesn't provide the proof since he's probably leaving it as an exercise. Here I'd like to provide the proof of this theorem that uses the version of the Schwartz kernel theorem in Euclidean space (c.f. [2] for the latter). Furthermore, I'd like to work through the slightly more general form of the theorem in which we allow for two different manifolds to be involved.

Theorem 2.18: *Suppose that M and N are smooth manifolds without boundaries and that $E \to M$ and $F \to N$ are two smooth vector bundles. Let $\pi_L : N \times M \to N$ and $\pi_R : N \times M \to M$ denote the maps $(p, q) \mapsto p$ and $(p, q) \mapsto q$ respectively, and let ΩM and ΩN denote the density bundles of M and N respectively. Then*

1. *for any sequentially continuous linear map $A : C_c^\infty(M; E) \mapsto \mathcal{D}'(N; F)$, there exists a continuous linear functional $K \in C_c^\infty(N \times M; \pi_L^*(F^* \otimes \Omega N) \otimes \pi_R^* E) \to \mathbb{C}$ that satisfies*

$$(2.19) \qquad \langle A\phi, \psi \rangle = \langle K, \pi_L^* \psi \otimes \pi_R^* \phi \rangle \qquad \forall \phi \in C_c^\infty(M; E) \;\; \forall \psi \in C_c^\infty(N; F^* \otimes \Omega N).$$

2. *for any continuous linear functional $K \in C_c^\infty(N \times M; \pi_L^*(F^* \otimes \Omega N) \otimes \pi_R^* E) \to \mathbb{C}$, the linear map $A : C_c^\infty(M; E) \mapsto \mathcal{D}'(N; F)$ given by (2.19) above is sequentially continuous.*

Proof: I will *not* use the Einstein summation convention in this proof. Let's start by proving (2). Pick any such K. First let's check that the associated map A maps between the claimed spaces. Take any $\phi \in C_c^\infty(M; E)$. We need to check that $A\phi$ is a continuous linear functional over $C_c^\infty(N; F^* \otimes \Omega N)$. Fix any coordinate chart $(V, (y^i))$ of N (V denotes the domain of the chart), any frame (f_j) of $F^* \otimes \Omega N$ over V, and any compact set $Q \subseteq V$. Let $\{(U_r, (x_r^i)) : r = 1, \dots, m\}$ be a finite cover of $\operatorname{supp} \phi$ by coordinate charts of M and fix a frame (e_l^r) over every U_r. Let $\mathcal{U} = \bigcup_{r=1}^m U_r$ and let $\{\rho_r \in C_c^\infty(\mathcal{U}) : r = 1, \dots, m\}$ be a partition of unity subordinate to $\{U_r\}$. Then for any $\psi \in C_c^\infty(N; F^* \otimes \Omega N)$ such that $\operatorname{supp} \psi \subseteq Q$,

$$\langle A\phi, \psi \rangle = \sum_{r=1}^{m} \langle K, \pi_L^* \psi \otimes \pi_R^*(\rho_r \phi) \rangle.$$

By the continuity of K, there exist $C, n > 0$ such that for each r,

$$|\langle K, \omega \rangle| \leq C \sum_j \sum_{|\alpha|,|\beta| \leq n} \sup \left| \partial_y^\alpha \partial_{x_r}^\beta \omega^j \right|$$

$$\forall \omega \in C_c^\infty(N \times M; \pi_L^*(F^* \otimes \Omega N) \otimes \pi_R^* E) : \operatorname{supp} \omega \subseteq \pi_L^{-1}[Q] \cap \pi_R^{-1}[\operatorname{supp} \rho_r],$$

where ω^j are the components of ω in the frame $(\pi_L^*(f_j), \pi_R^*(e_l^r))$. Hence

$$|\langle A\phi, \psi \rangle| \leq C \sum_{r=1}^{m} \sum_{j=1}^{\operatorname{rank} F} \sum_{l=1}^{\operatorname{rank} E} \sup \left| \partial_y^\alpha \psi^j \cdot \partial_{x_r}^\beta (\rho_r \phi^l) \right|$$

where ψ^j are the components of ψ in the frame (f_j) and $\rho_r \phi^l$ are the components of $\rho_r \phi$ in the frame (e_l^r). Using that the $\partial_{x_r}^\beta(\rho_r \phi^l)$ are bounded, its not hard to see that this shows that $A\phi$ is indeed continuous. We leave it to the reader to show that A itself is sequentially continuous since the details involved are very similar to what we just did.

Now let's prove (1). Fix any such A, and let's construct the desired K. Resetting our notation, let $\{(U_r, (x_r^i)) : r = 1,2, \dots\}$ be a countable cover of M and let $\{\rho_r \in C_c^\infty(M) : r = 1,2, \dots\}$ be a partition of unit subordinate to this cover. For each r, let (e_μ^r) be a frame for E over U_r. Similarly, let $\{(V_s, (y_s^j) : s = 1,2, \dots\}$ be a countable cover of N and let $\{\sigma_s \in C_c^\infty(N) : s = 1,2, \dots\}$ be a partition of unit subordinate to this cover. For each s, let (f_ν^s) be a frame for $F^* \otimes \Omega N$ over V_s. We have that for any $\phi \in C_c^\infty(M; E)$ and any $\psi \in C_c^\infty(N; F^* \otimes \Omega N)$

$$\langle A\phi, \psi \rangle = \sum_{r,s=1}^{\infty} \langle A(\rho_r \phi), \rho_s \psi \rangle.$$

Now each term in this sum can be written as

$$\sum_{\mu=1}^{\operatorname{rank} E} \sum_{\nu=1}^{\operatorname{rank} F} \langle A\left[(\rho_r \phi^\mu) e_\mu^r\right], (\sigma_s \psi^\nu) f_\nu^s \rangle.$$

where $\rho_s\phi^\mu$ and $\sigma_s\psi^\nu$ are the components of $\rho_r\phi$ and $\sigma_s\psi$ with respect to (e_μ^r) and (f_ν^s). Fix any r, s, μ, ν and consider the linear map $A_{r,s,\mu,\nu} : \mathcal{D}'(U_r) \to \mathcal{D}'(V_s)$, where $\mathcal{D}'(U_r)$ and $\mathcal{D}'(V_s)$ are the ordinary spaces of distributions as defined in [2], given by

$$\langle A_{r,s,\mu,\nu}u, v\rangle = \langle A\left[ue_\mu^r\right], vf_\nu^s\rangle \qquad \forall u \in C_c^\infty(U_r) \;\; \forall v \in C_c^\infty(V_s)$$

(we leave it to the reader to show that $A_{r,s,\mu,\nu}$ maps between the claimed spaces). Observe that if we took a sequence $\{u_t\} \subseteq C_c^\infty(U_r)$ that goes to zero as $t \to \infty$ as defined in [2], then $(\rho_r u_t)e_\mu^r$ would also go to zero as defined in Definition 2.2 above. Hence by the continuity of A the above equation tells us that $\langle A_{r,s,\mu,\nu}u_t, v\rangle \to 0$ as $t \to \infty$ for any $v \in C_c^\infty(V_s)$. In other words, $A_{r,s,\mu,\nu}$ is also sequentially continuous. Thus by the version of the Schwartz kernel theorem proved in [2], $A_{r,s,\mu,\nu}$ has a Schwartz kernel $Q_{r,s,\mu,\nu} \in \mathcal{D}'(V_s \times U_r)$ satisfying

$$\langle A_{r,s,\mu,\nu}u, v\rangle = \langle Q_{r,s,\mu,\nu}, v \otimes u\rangle \qquad \forall u \in C_c^\infty(U_r) \;\; \forall v \in C_c^\infty(V_s).$$

Let $K_{r,s,\mu,\nu} : C_c^\infty(N \times M; \pi_L^*(F^* \otimes \Omega N) \otimes \pi_R^* E) \to \mathbb{C}$ be given by

$$\langle K_{r,s,\mu,\nu}, w\rangle = \langle Q_{r,s,\mu,\nu}, \rho_r\sigma_s w^{\mu,\nu}\rangle$$

where $w^{\mu,\nu}$ is the component of w in $\pi^* f_\nu^s \otimes \pi^* e_\mu^r$ with respect to the full frame $(\pi_L^* f_{\nu'}^s \otimes \pi_R^* e_{\mu'}^r)$. We leave it to the reader to show that $K_{r,s,\mu,\nu}$ is continuous (hint: show sequential continuity). It's not hard to see that

$$\langle A\left[(\rho_r\phi^\mu)e_\mu^r\right], (\rho_s\psi^\nu)f_\nu^s\rangle = \langle K_{r,s,\mu,\nu}, \pi^*\psi \otimes \pi^*\phi\rangle.$$

Hence we get that

$$\langle A\phi, \psi\rangle = \sum_{r,s=1}^{\infty} \sum_{\mu=1}^{\operatorname{rank} E} \sum_{\nu=1}^{\operatorname{rank} F} \langle K_{r,s,\mu,\nu}, \pi_L^*\psi \otimes \pi_R^*\phi\rangle.$$

Observe that the (distributional) support of each $K_{r,s,\mu,\nu}$ is contained in $\operatorname{supp}\rho_r \times \operatorname{supp}\rho_s$. By construction, the latter is locally finite as r and s vary from 1 to ∞. It's not hard to see that this implies that the sum of distributions $K = \sum_{r,s=1}^{\infty}\sum_{\mu=1}^{\operatorname{rank} E}\sum_{\nu=1}^{\operatorname{rank} F} K_{r,s,\mu,\nu}$ is well defined, continuous, and furthermore

$$\langle A\phi, \psi\rangle = \langle K, \pi_L^*\psi \otimes \pi_R^*\phi\rangle.$$

This proves (1).

■

If one is not interested in the case when the vector bundles E, F are present, then one has the following result whose proof is essentially the same as the theorem above.

Theorem 2.20: *Suppose that M and N are smooth manifolds without boundaries. Let $\pi_L : N \times M \to N$ and $\pi_R : N \times M \to M$ denote the maps $(p, q) \mapsto p$ and $(p, q) \mapsto q$ respectively, and let ΩM and ΩN denote the density bundles of M and N respectively. Then*

1. *for any sequentially continuous linear map* $A : C_c^\infty(M) \mapsto \mathcal{D}'(N)$, *there exists a continuous linear functional* $K \in C_c^\infty\big(N \times M; \pi_L^*(\Omega N)\big) \to \mathbb{C}$ *that satisfies*

$$(2.21) \qquad \langle A\phi, \psi\rangle = \langle K, \pi_L^*\psi \otimes \pi_R^*\phi\rangle \qquad \forall \phi \in C_c^\infty(M) \quad \forall \psi \in C_c^\infty(N; \Omega N).$$

2. *For any continuous linear functional* $K \in C_c^\infty\big(N \times M; \pi_L^*(\Omega N)\big) \to \mathbb{C}$, *the linear map* $A : C_c^\infty(M) \mapsto \mathcal{D}'(N)$ *given by* (2.21) *above is sequentially continuous.*

2.3 Smooth Schwartz Kernels

Hintz in his notes discusses the form of Schwartz Kernels when the kernels are smooth. In these notes I'd like to elaborate on that with the additional relaxation of not requiring the two manifolds to be identical.

Suppose that M and N are smooth manifolds without boundaries of dimensions m and n respectively, and that $E \to M$ and $F \to N$ are smooth vector bundles. Let $\pi_L : N \times M \to M$ and $\pi_R : N \times M \to N$ denote the maps $(p, q) \mapsto p$ and $(p, q) \mapsto q$ respectively and let ΩM, ΩN, and $\Omega(N \times M)$ denote the density bundles of M, N, and $N \times M$ respectively.

If the Schwartz Kernel of a continuous linear operator $A : C_c^\infty(M; E) \to D'(N; F)$ is smooth, then mathematicians will say that it is of the form

$$(2.22) \qquad K \in C^\infty\big(N \times M; \pi_L^* F \otimes \pi_R^*(E^* \otimes \Omega M)\big).$$

In this note, I'd like to discuss how and why such K's are canonically identified with being continuous linear functionals of the form $C_c^\infty(N \times M; \pi_L^*(F^* \otimes \Omega N) \otimes \pi_R^* E) \to \mathbb{C}$. To do this, let's define a natural map $\mathcal{C}$ that "combines" sections of $\pi_L^*(F^* \otimes \Omega M) \otimes \pi_R^* E$ and $\pi_L^* F \otimes \pi_R^*(E^* \otimes \Omega M)$ and outputs a section of $\Omega(N \times M)$.

We do this locally. Fix any point $(p, q) \in N \times M$ where $p \in N$ and $q \in M$. Let $(V, (y^j))$ and $(U, (x^i))$ be coordinates of N and M in neighborhoods of p and q respectively (U and V denote the coordinates' domains), which of course generate coordinates (y^j, x^i) of $N \times M$. Let $\big(e_\mu\big)$, $(e^{*\mu})$, (f_ν), $(f^{*\nu})$ be frames for E, E^*, F, F^* over U and V accordingly. Then, over $V \times U$ we define

$$(2.23)$$

$$\mathcal{C}\left(a_\nu^\mu \pi_L^*(f^{*\nu} \otimes |dy^1 \wedge \dots \wedge dy^n|) \otimes \pi_R^* e_\mu, \qquad b_\mu^\nu \pi_L^* f_\nu \otimes \pi_R^*(e^{*\mu} \otimes |dx^1 \wedge \dots \wedge dx^m|)\right)$$
$$= a_\nu^\mu b_\mu^\nu |dy^1 \wedge \dots \wedge dy^n \wedge dx^1 \wedge \dots \wedge dx^m|.$$

Let's check that this is well defined. Suppose that $(\tilde{U}, (\tilde{x}^i))$, $(\tilde{V}, (\tilde{y}^j))$, $\big(\tilde{e}_\mu\big)$, $(\tilde{e}^{*\mu})$, $\big(\tilde{f}_\nu\big)$, $\big(\tilde{f}^{*\nu}\big)$ are similar quantities as above. We have to show that over $(V \times U) \cap \big(\tilde{V} \times \tilde{U}\big)$ the above quantity is equal to

$$(2.24) \qquad \tilde{a}_\nu^\mu \tilde{b}_\mu^\nu |d\tilde{y}^1 \wedge \dots \wedge d\tilde{y}^n \wedge d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^m|$$

where $\tilde{a}_\nu^\mu$ and $\tilde{b}_\mu^\nu$ represent the components of the same sections involved in the previous equation but with respect to the frames that have "tildes" over them. Let $\mathcal{A}$ and $\mathcal{B}$ be the matrices in the relations

$$\tilde{e}_\mu = \mathcal{A}_\mu^\lambda e_\lambda \qquad \text{and} \qquad \tilde{f}_\nu = \mathcal{B}_\nu^\eta f_\eta,$$

whose inverses recall satisfy

$$\tilde{e}^{*\mu} = (\mathcal{A}^{-1})_\lambda^\mu e^{*\lambda} \qquad \text{and} \qquad \tilde{f}^{*\nu} = (\mathcal{B}^{-1})_\eta^\nu f^{*\eta}.$$

We leave it to the reader to show that (with lack of sufficient indices, I put hats over the indices in the second quantity below)

$$a_\nu^\mu = \pi_L^*\left[(B^{-1})_\nu^\eta \left|\det\frac{\partial\tilde{y}}{\partial y}\right|\right]\pi_R^*\left[A_\lambda^\mu\right]\tilde{a}_\eta^\lambda \qquad \text{and} \qquad b_{\hat{\mu}}^{\hat{\nu}} = \pi_L^*\left[B_{\hat{\eta}}^{\hat{\nu}}\right]\pi_R^*\left[(A^{-1})_{\hat{\mu}}^{\hat{\lambda}}\left|\det\frac{\partial\tilde{x}}{\partial x}\right|\right]\tilde{b}_{\hat{\lambda}}^{\hat{\eta}}.$$

Hence using this and the transformation law from $|dy^1 \wedge \dots \wedge dy^n \wedge dx^1 \wedge \dots \wedge dx^m|$ to $|d\tilde{y}^1 \wedge \dots \wedge d\tilde{y}^n \wedge d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^m|$, we get that (2.23) is equal to (the following is one big quantity written in two lines)

$$\left(\pi_L^*\left[(B^{-1})_\nu^\eta\left|\det\frac{\partial\tilde{y}}{\partial y}\right|\right]\pi_R^*\left[A_\lambda^\mu\right]\tilde{a}_\eta^\lambda\right)\left(\pi_L^*\left[B_{\hat{\eta}}^{\hat{\nu}}\right]\pi_R^*\left[(A^{-1})_{\hat{\mu}}^{\hat{\lambda}}\left|\det\frac{\partial\tilde{x}}{\partial x}\right|\right]\tilde{b}_{\hat{\lambda}}^{\hat{\eta}}\right)$$

$$\pi_L^*\left[\left|\det\frac{\partial y}{\partial\tilde{y}}\right|\right]\pi_R^*\left[\left|\det\frac{\partial x}{\partial\tilde{x}}\right|\right]|d\tilde{y}^1 \wedge \dots \wedge d\tilde{y}^n \wedge d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^m|.$$

Because there are so many matrices and their inverses involved here, essentially everything cancels out to give (2.24). Hence $\mathcal{C}$ is indeed well defined.

Having established this, we are ready to discuss how (2.22) is associated to a linear functional of the form we were mentioning earlier. Take any K as in (2.22). For any $\phi \in C_c^\infty(N \times M; \pi_L^*(F^* \otimes \Omega N) \otimes \pi_R^* E)$ we set

$$\langle K, \phi\rangle = \int_{N\times M} \mathcal{C}(K, \phi).$$

This obviously linear in ϕ. Let's see why it's also continuous. Observe for instance that if we took a compact subset $Q \subseteq V \times U$ of the above coordinates, the above equation in local coordinates would look like

$$\int_{V\times U} K_\nu^\mu \phi_\mu^\nu \, dy^1 \dots dy^n dx^1 \dots dx^m.$$

where the ϕ_μ^ν are supported in Q. It should be clear from here that the action of K on ϕ is indeed continuous.

3 References

Citation of additional works:

1. Hörmander, L. (1985, 2009). *The Analysis of Linear Partial Differential Operators*. Berlin Heidelberg: Springer-Verlag.
2. Friedlander, G., & Joshi, M. (1998). *Introduction to the Theory of Distributions* (2nd ed.). Cambridge University Press.