The Non-Abelian X-Ray Transform on Asymptotically Hyperbolic Spaces

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Abstract

In this paper we formulate and prove a gauge equivalence for connections and Higgs fields of suitable regularity that are mapped to the same function under the non-abelian X-ray transform on nontrapping asymptotically hyperbolic (AH) spaces with negative curvature and no nontrivial twisted conformal Killing tensor fields of suitable decay. If one furthermore fixes such a connection with zero curvature, a corollary provides an injectivity result for the non-abelian X-ray transform over Higgs fields.

Contents

1 Conventions/notations

In this paper we employ the following conventions/notations:

1. We employ the Einstein summation convention.
2. The dimension of our manifold will be $n+1$, and we'll denote all indices related to its dimension by $0, 1, \ldots, n$. When using the Einstein summation convention on indices related to the manifold's dimension, we employ the convention that Latin indices can sum from 0 to n while Greek indices can only sum from 1 to n.
3. Whenever we say "smooth," we mean "C^∞." All diffeomorphisms are smooth.
4. If $\pi : E \to N$ is a vector bundle over a manifold N and $S \subseteq N$ is a subset of N, then we let $E|_S$ denote the restriction of E to the fibers over S (i.e. more precise notation would be $E|_{\pi^{-1}(S)}$).
5. Continuing off of point 4), we write $C^\infty(N; E)$ for smooth sections of E (i.e. not simply smooth maps from N to E).
6. We denote the geodesic vector field over the tangent and cotangent bundles (i.e. infinitesimal generator of the geodesic flow) by X. We will also let X denote its restriction to the unit tangent and cotangent bundles SM and S^*M (described below), which makes sense because X is tangent to it.

2 Introduction

2.1 Motivation

In this section we provide motivation for the non-abelian X-ray transform - the object of our interest, delaying precise definitions for a later section. This transform is a generalization of the so-called "scalar X-ray transform," the latter of which is used in reconstructing images of the internals of patients after irradiating them with X-rays at various angles. The typical mathematical problem for the scalar X-ray transform is the following: suppose that we have a bounded subset $D \subseteq \mathbb{R}^n$ with smooth boundary and a smooth function $\phi : D \to (0, \infty)$ over it. In our analogy, D represents the shape of our patient and ϕ the body's X-ray absorption coefficient at various points. Suppose we have a parametrized line $l(t)$ that enters D at $t = 0$

and leaves at $t = t_{\text{exit}}$, which represents the motion of an X-ray moving through the body. The ray's intensity $I(t)$ along this path decays according to the law

$$\frac{dI}{dt} = -\phi I, \qquad I(0) = I_0, \tag{2.1}$$

where I_0 represents the initial intensity of the ray. We record $I(t_{\text{exit}})$ (i.e. the intensity of the ray when it exits) and then repeat this procedure for all possible lines l that pass through D, using the same value for I_0 every time. The inverse problem is then to recover the coefficient ϕ from knowing such data, which is equivalent to recovering a gray-scale image of the patient's internals. Immediately we may note that a necessary condition for this to be possible to do is that two different ϕ's cannot generate the same observation data. If this is the case, we say that the scalar X-ray transform that takes ϕ to the observed data is "injective."

The non-abelian X-ray transform is defined similarly, except that we turn (2.1) into a system of equations by letting I by a column vector and ϕ a square matrix. The question then is whether or not it's possible to recover ϕ from the collected data in this case as well, or in other words if the operator involved is injective. One application of this is in the recently introduced *polarimetric neutron tomography* which attempts to reconstruct the structure of magnetic fields inside materials after sending neutron beams through them - see for instance [18] and [7]. We will mention a few more applications of this problem with references in Section 2.5 below.

We will actually be interested in a more sophisticated generalization of the transform, for instance by allowing the paths "l" to be geodesics with respect to some Riemannian metric g on D where D is now a smooth manifold. Furthermore, we will formulate I and ϕ to be endomorphism fields over a smooth vector bundle over M and formulate the time derivative to be a connection (which may also be unknown) in the direction of the curve's velocity.

2.2 Asymptotically Hyperbolic Spaces

We begin by introducing the geometry on which our transform will be defined. In this paper we let $\overline{M}$ be a compact connected smooth manifold with smooth boundary of dimension $n+1$ with $n \geq 1$, whose interior we denote by M. We fix a boundary defining function $\rho : M \to [0, \infty)$ (i.e. ρ is smooth, $\rho = 0$ only on $\partial\overline{M}$, and $d\rho|_{\partial\overline{M}} \neq 0$). We also fix an **asymptotically hyperbolic (AH) Riemannian metric** g on M, which is defined as a metric such that the tensor $\overline{g} = \rho^2 g$ extends to a smooth Riemannian metric on all of $\overline{M}$ with $|d\rho|^2_{\rho^2 g} \equiv 1$ along $\partial\overline{M}$. The boundary of $\partial\overline{M}$ is thought of as the "infinity" where the metric g blows up. Hence recalling that hyperbolic space has constant sectional curvature -1, the known fact that the sectional curvatures of g tend to -1 as one approaches $\partial\overline{M}$ explains why such metrics are given the name "AH.[1]"

In fact, the Poincaré ball model of hyperbolic space is the archetypical example of an AH

[1]More generally, the sectional curvatures approaches $-|d\rho|^2$ restricted to the boundary - see [9] for a precise statement.

space. It is given by $\overline{M} = \{|x| \leq 1\} \subseteq \mathbb{R}^{n+1}$, where $|x|$ denotes the Euclidean length, and

$$g = 4\frac{(dx^1)^2 + \ldots + (dx^{n+1})^2}{\left(1 - |x|^2\right)^2}.$$

Indeed if one takes the boundary defining function $\rho = 1 - |x|^2$, then an elementary exercise shows that $|d\rho|^2_{\rho^2 g} \equiv 1$ along $\{|x| = 1\}$.

Taking our general AH space, we note that we can always construct boundary coordinates of $\overline{M}$ of the form $(\rho, y^1, \ldots, y^n)$ in which the metric g takes the following convenient form. Let $\varepsilon > 0$ be such that the flowout of the gradient of ρ with respect to the (smooth) metric $\overline{g} = \rho^2 g$ is a diffeomorphism from $(0, \varepsilon) \times \partial\overline{M}$ onto a collar neighborhood $\mathcal{C}_\varepsilon$ of $\partial\overline{M}$. Then, fixing coordinates $(y^1, \ldots, y^n)$ of $\partial\overline{M}$, this flowout provides us boundary coordinates (ρ, y^μ) of $\overline{M}$ in which the metric takes the form

$$g = \frac{d\rho^2 + h_{\mu\nu} dy^\mu dy^\nu}{\rho^2}. \tag{2.2}$$

We call such boundary coordinates (ρ, y^μ) **asymptotic boundary normal coordinates**.

By Proposition 1.8 in [27], AH spaces are complete. Furthermore, in some cases we will assume that g is also **nontrapping** which means that for any complete g-geodesic $\gamma : (-\infty, \infty) \to M$, $\liminf_{t \to \pm\infty} \rho(\gamma(t)) = 0$. Intuitively speaking, this condition requires that γ eventually "escapes to infinity."

2.3 Results

We now state our results. As mentioned in Section 2.2 above we assume in this paper that $\left(M \subseteq \overline{M}, g\right)$ is a connected AH space and that ρ is a boundary defining function. Throughout this paper we also assume that we have a smooth complex d-dimensional vector bundle $\mathcal{E}$ over $\overline{M}$ equipped with a smooth Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$. Moreover, we assume that we have a smooth section of the endomorphism bundle $\Phi : \overline{M} \to \operatorname{End}(\mathcal{E})$ that is skew-Hermitian with respect to $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ (i.e. $\langle \Phi u, v \rangle_{\mathcal{E}} = -\langle u, \Phi v \rangle_{\mathcal{E}}$). Lastly, we assume that we have a unitary smooth connection $\nabla^{\mathcal{E}}$ in $\mathcal{E}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ - meaning that

$$V\langle u, v \rangle_{\mathcal{E}} = \langle \nabla^{\mathcal{E}}_V u, v \rangle_{\mathcal{E}} + \langle u, \nabla^{\mathcal{E}}_V v \rangle_{\mathcal{E}},$$

when V is any smooth vector field over $\overline{M}$ and u, v are any smooth sections of $\mathcal{E}$.

The analog of the (2.1) that we will be considering is the following. Take any unit-speed complete geodesic $\gamma : (-\infty, \infty) \to M$ such that $\liminf_{t \to \pm\infty} \rho(\gamma(t)) = 0$. It follows from Lemma 2.3 in [12] that the limit of $\gamma(t)$ in $\overline{M}$ exists as $t \to \pm\infty$. Let $u : (-\infty, \infty) \to \mathcal{E}$ be a smooth section along γ solving the following initial value problem:

$$\nabla^{\mathcal{E}}_{\dot{\gamma}(t)} u(t) + \Phi(\gamma(t)) u(t) = 0, \qquad \lim_{t \to -\infty} u(t) = e, \tag{2.3}$$

where e is any element in $\mathcal{E}_{x_0}$ where x_0 is the limit of $\gamma(t)$ in $\overline{M}$ as $t \to -\infty$. The data point that we "record" is

$$\lim_{t \to \infty} u(t). \tag{2.4}$$

The question that we are interested then becomes whether we can recover Φ and $\nabla^{\mathcal{E}}$ from the data recorded for all such possible pairs γ and e.

A bit of vocabulary: (2.3) is a type of differential equation called a **transport equation**, and Φ is called a **Higgs field**. Going from the pair $\left(\nabla^{\mathcal{E}}, \Phi\right)$ to the map that takes every (γ, e) as above to its associated data (2.4) is called the **non-abelian X-ray transform**, of which we give a more precise definition in Section 2.4 below.

To make rigorous sense of our problem however, we need to establish the well-definedness of the solution to (2.3) and the data (2.4). Considering that we're making use of the values of the solution to (2.3) at plus or minus infinities, we accomplish this by imposing decay conditions on both Φ and the connection symbols of $\nabla^{\mathcal{E}}$:

Lemma 2.5. *Suppose that*[2] $\Phi \in \rho^\infty C^\infty\left(\overline{M}; \operatorname{End}_{\mathrm{Sk}}\mathcal{E}\right)$ *and that the connection symbols of* $\nabla^{\mathcal{E}}$ *are in* $\rho^\infty C^\infty\left(\overline{M}; \mathbb{R}\right)$ *in any boundary coordinates and frame (for* $\mathcal{E}$*). Then for any complete geodesic* $\gamma : (-\infty, \infty) \to M$ *and any* $e \in \mathcal{E}_{x_0}$ *where* $x_0 = \lim_{t\to-\infty} \gamma(t)$ *in* $\overline{M}$*, the solution to (2.3) exists and so does the limit (2.4).*

Intuitively speaking, in the above lemma we require that Φ and $\nabla^{\mathcal{E}}$'s connection symbols vanish to infinite order at "infinity" (i.e. $\{\rho = 0\} = \partial\overline{M}$). For future use, we remark that the above lemma and its proof work equally well if one changes "$t \to \pm\infty$" to "$t \to \mp\infty$" in its statement and in (2.3) and (2.4).

The following is our main result. To state it, we use the notion of *twisted conformal Killing tensor fields* (CKTs for short) that decay to order ${}^{(n+1)}/_2$ or faster at infinity. We define these at the end of Section 5.2 below and is a technical assumption on g and $\nabla^{\mathcal{E}}$ that we need to make the vertical Fourier analysis in its proof to work.

Theorem 2.6. *Assume that* (M, g) *is nontrapping and that the sectional curvatures of* g *are negative. Suppose that* $\Phi \in \rho^\infty C^\infty\left(\overline{M}; \operatorname{End}_{\mathrm{Sk}}\mathcal{E}\right)$ *and that the connection symbols of* $\nabla^{\mathcal{E}}$ *are in* $\rho^\infty C^\infty\left(\overline{M}; \mathbb{R}\right)$ *in any boundary coordinates and frame. Suppose also that there are no nontrivial twisted CKTs that decay to order* ${}^{(n+1)}/_2$ *or faster at infinity. Now, suppose that we have another pair* $\widetilde{\Phi}$ *and* $\widetilde{\nabla}^{\mathcal{E}}$ *satisfying the same conditions as* Φ *and* $\nabla^{\mathcal{E}}$*. Lastly, suppose that the data (2.4) for all possible* γ *and* e *as above are the same for (2.3), and (2.3) with* Φ *and* $\nabla^{\mathcal{E}}$ *replaced by* $\widetilde{\Phi}$ *and* $\widetilde{\nabla}^{\mathcal{E}}$ *respectively. Then there exists an invertible* $Q \in C^\infty\left(\overline{M}; \operatorname{End}\mathcal{E}\right)$ *such that* $Q|_{\partial\overline{M}} = \mathrm{id}$ *with* $(Q - \mathrm{id}) \in \rho^\infty C^\infty\left(\overline{M}\right)$*, and over* $\overline{M}$ *satisfies*

$$\widetilde{\nabla}^{\mathcal{E}} = Q^{-1}\nabla^{\mathcal{E}}Q, \qquad \widetilde{\Phi} = Q^{-1}\Phi Q. \tag{2.7}$$

Note: The notation $\widetilde{\nabla}^{\mathcal{E}} = Q^{-1}\nabla^{\mathcal{E}}Q$ means

$$\widetilde{\nabla}^{\mathcal{E}}_v e = Q^{-1}\nabla^{\mathcal{E}}_v(Qe)$$

for any tangent vector $v \in TM$ and any section $e \in C^\infty\left(\overline{M}; \mathcal{E}\right)$.

[2]The notation $\rho^\infty C^\infty(\ldots)$ means $\bigcap_{N\in\mathbb{R}} \rho^N C^\infty(\ldots)$.

In other words, the theorem states that we cannot recover the connection and Higgs field from simply knowing the data (2.4) for all possible (γ, e) as above because such data can come from two distinct $\left(\nabla^{\mathcal{E}}, \Phi\right)$ and $\left(\widetilde{\nabla}^{\mathcal{E}}, \widetilde{\Phi}\right)$. However if that is the case, then there is an endomorphism field Q that relates the two in the **gauge** relation (2.7).

The relation (2.7) may look mysterious at first, so let us provide intuition for it. The proof of Theorem 2.6 essentially begins with taking (2.3) and rewriting it in terms of endomorphism fields $U : \mathbb{R} \to \operatorname{End} \mathcal{E}$ in a way that it encodes the same data:

$$\nabla^{\mathrm{En}}_{\dot{\gamma}(t)} U(t) + \Phi(\gamma(t)) U(t) = 0, \qquad \lim_{t \to -\infty} U(t) = \mathrm{id},$$

where "∇^{En}" is a natural connection on the space of endomorphism fields generated by $\nabla^{\mathcal{E}}$. Then we ask the following question. Suppose that we have a connection $\nabla^{\mathcal{E}}$, Higgs field Φ, and the data (2.4) that it generates as above. How can we "come up" with another pair $\widetilde{\nabla}^{\mathcal{E}}$ and $\widetilde{\Phi}$ that generates the same data set? To do this, we take an arbitrary endomorphism field Q and manipulate the above equality as follows:

$$\begin{aligned}
\nabla^{\mathrm{En}}_{\dot{\gamma}}\left(QQ^{-1}U\right) + \Phi\left(QQ^{-1}U\right) &= 0, && \text{since } QQ^{-1} = \mathrm{id}, \\
\nabla^{\mathrm{En}}_{\dot{\gamma}}(Q) Q^{-1} U + Q \nabla^{\mathrm{En}}_{\dot{\gamma}}\left(Q^{-1}U\right) + \Phi\left(QQ^{-1}U\right) &= 0, && \text{product rule}, \\
Q^{-1} \nabla^{\mathrm{En}}_{\dot{\gamma}}(Q) \widetilde{U} + \nabla^{\mathrm{En}}_{\dot{\gamma}} \widetilde{U} + \left(Q^{-1}\Phi Q\right)\widetilde{U} &= 0, && \text{multiply by } Q^{-1} \text{ and set } \widetilde{U} = Q^{-1}U, \\
\widetilde{\nabla}^{\mathrm{En}}_{v} \widetilde{U} + \widetilde{\Phi}\widetilde{U} &= 0,
\end{aligned}$$

where we set $\widetilde{\nabla}^{\mathrm{En}}_{v} \widetilde{U} := \nabla^{\mathrm{En}}_{v} \widetilde{U} + Q^{-1} \nabla^{\mathrm{En}}_{v}(Q) \widetilde{U}$ and $\widetilde{\Phi} := Q^{-1}\Phi Q$. It's quick to see that $\widetilde{\nabla}^{\mathrm{En}}_{v}$ and $\widetilde{\Phi}$ will generate the same data set (2.4) if $Q|_{\partial \overline{M}} = \mathrm{id}$ and that this Q satisfies (2.7). The main of point of Theorem 2.6 is that this is the only way that we can produce another pair $\widetilde{\nabla}^{\mathcal{E}}$ and $\widetilde{\Phi}$ with the same data set when the Higgs fields are skew symmetric. This example is illustrative in the sense that from the above we see that the gauge Q is given by

$$Q = U \widetilde{U}^{-1}. \tag{2.8}$$

This equation will in fact be the idea of how we will come up with the gauge in the proof of Theorem 2.6.

To elaborate more on the outline of the proof, we will show that $Q - \mathrm{id}$ satisfies a specific transport equation of the form

$$\nabla^{\pi^* \mathrm{En}}_{X} W + \Psi W = f$$

on the 0-cosphere bundle where X is the geodesic vector field (c.f. (5.52) below). The right-hand side f will turn out to have rapid decay at infinity and Fourier modes of order no bigger than one with respect to the vertical Laplacian. In Section 5.3 we will prove a regularity theorem for transport equations that will imply that W also has rapid decay at infinity. In Section 5.2 we conduct a Fourier study of transport equations that, combined with the just mentioned result, will imply that W, and hence Q, have Fourier degree zero (i.e. are of the form $C^\infty\left(\overline{M}; \operatorname{End} \mathcal{E}\right)$). From there it will quickly follow that Q satisfies the conclusions in Theorem 2.6.

We end this subsection by pointing out that in the special case of when the connection is known and has zero curvature (see Section 4.6 below for the latter), then it is possible to recover the Higgs field. Here is the precise statement:

Corollary 2.9. *Assume that (M, g) is nontrapping, the sectional curvatures of g are negative, and that the curvature of $\nabla^{\mathcal{E}}$ is zero everywhere. Suppose that $\Phi \in \rho^\infty C^\infty\big(\overline{M}; \mathrm{End}_{\mathrm{Sk}}\mathcal{E}\big)$ and that the connection symbols of $\nabla^{\mathcal{E}}$ are in $\rho^\infty C^\infty\big(\overline{M}; \mathbb{R}\big)$ in any boundary coordinates and frame. Suppose also that there are no nontrivial twisted CKTs that decay to order ${}^{(n+1)}/_2$ or faster at infinity. Now, suppose that we have another $\widetilde{\Phi}$ satisfying the same decay condition as Φ. Lastly, suppose that the data (2.4) for all possible γ and e as above are the same for (2.3), and (2.3) with Φ replaced by $\widetilde{\Phi}$. Then $\Phi = \widetilde{\Phi}$.*

A simple example of when the corollary applies is if $\overline{M}$ is a subset of $\mathbb{R}^{n+1}$, $\mathcal{E} = \overline{M} \times \mathbb{C}^d$ is the trivial bundle whose sections we write as column vectors, and the connection $\nabla^{\mathcal{E}}$ is simply given by $\nabla^{\mathcal{E}}_v u = \big[v(u^1), \ldots, v\big(u^d\big)\big]$.

2.4 Non-Abelian X-Ray Transform

We mention a way to formalize the operator that takes (∇, Φ) to the map taking pairs (γ, e) as above to (2.4) using concepts that we introduce in Sections 4.1, 4.2, and 4.5 below. We will not make use of this formulation, and only the material up to (2.9) and the two sentences afterwards here will be used later in the paper. Throughout this section we assume that (M, g) is nontrapping, $\Phi \in \rho^\infty C^\infty\big(\overline{M}; \mathrm{End}\,\mathcal{E}\big)$, and that the connection symbols of $\nabla^{\mathcal{E}}$ are in $\rho^\infty C^\infty\big(\overline{M}; \mathbb{R}\big)$ in any boundary coordinates and frame.

Consider the cotangent and b-cotangent bundles T^*M and ${}^bT^*\overline{M}$ respectively, and their unit cosphere bundles S^*M and ${}^bS^*\overline{M}$ respectively. Suppose $(\rho, y^1, \ldots, y^n)$ are boundary coordinates of $\overline{M}$ and consider the frame $({}^{d\rho}/_{\rho}, dy^1, \ldots, dy^n)$ spanning covectors in T^*M. On page 2865 of [12] the authors remind the reader that this extends to the boundary to become a smooth frame of ${}^bT^*\overline{M}$ and that furthermore if $\zeta = \eta_0 {}^{d\rho}/_{\rho} + \eta_\mu dy^\mu \in {}^bT^*\overline{M}\big|_{\partial\overline{M}}$ is over the boundary, then the map

$$\eta_0 \frac{d\rho}{\rho} + \eta_\mu dy^\mu \longmapsto \eta_0$$

is well defined (i.e. independent of the coordinates (ρ, y^μ) that we choose). The boundary of the unit cosphere bundle ${}^bS^*\overline{M} \subseteq {}^bT^*\overline{M}$ turns out to have the following two components:

$$\begin{aligned} \partial_-{}^bS^*\overline{M} &= \big\{\zeta \in {}^bT^*\overline{M}\big|_{\partial\overline{M}} : \eta_0 = 1\big\} && \text{called the “incoming boundary,”} \\ \partial_+{}^bS^*\overline{M} &= \big\{\zeta \in {}^bT^*\overline{M}\big|_{\partial\overline{M}} : \eta_0 = -1\big\} && \text{called the “outgoing boundary.”} \end{aligned}$$

Let $\pi : S^*M \to M$ and $\pi_b : {}^bS^*\overline{M} \to \overline{M}$ denote the natural projection maps. Recall that any unit-speed geodesic $\gamma : (-\infty, \infty) \to M$ is the image under π of an integral curve $\sigma : (-\infty, \infty) \to S^*M$ of the geodesic vector field X. Letting X_b denote the pushforward of this field onto ${}^bS^*M\big|_M$ via the canonical identification between T^*M and ${}^bT^*\overline{M}\big|_M$, we have that γ is the image under π_b of an integral curve $\sigma_b : (-\infty, \infty) \to {}^bS^*M\big|_M$ of X_b. It follows from the proof of Lemma 2.1 in [12] that the limit of any such curve exists in ${}^bS^*\overline{M}$:

$$\begin{aligned} \lim\nolimits_{t\to-\infty} \sigma_b &= \partial_-{}^bS^*\overline{M}, \\ \lim\nolimits_{t\to\infty} \sigma_b &= \partial_+{}^bS^*\overline{M}. \end{aligned} \tag{2.10}$$

Intuitively, the first limit here can be thought of as the “initial velocity” of the geodesic as it “enters” the AH space, while the second its “exit velocity” as it “leaves.” Conversely, every

$\zeta \in \partial_- {}^bS^*\overline{M}$ (resp. $\zeta \in \partial_+ {}^bS^*\overline{M}$) is the limit in ${}^bS^*\overline{M}$ of a unique (up to reparameterization) such curve σ_b as $t \to -\infty$ (resp. $t \to \infty$).

Hence we may define the map

$$T^{(\nabla^{\mathcal{E}},\Phi)} : \pi^*\mathcal{E}|_{\partial_- {}^bS^*\overline{M}} \to \pi^*\mathcal{E}|_{\partial_+ {}^bS^*\overline{M}}$$

as follows. Take any $e \in \pi^*\mathcal{E}|_{\partial_- {}^bS^*\overline{M}}$ whose based point we denote by $\zeta \in \partial_- {}^bS^*\overline{M}$. Let σ_b be an integral curve of X_b with $\zeta = \lim_{t\to-\infty} \sigma_b$ and let $\zeta_{\text{exit}} = \lim_{t\to\infty} \sigma_b$. Take the geodesic $\gamma = \pi_b \circ \sigma_b$ and let u be the solution to (2.3) where we let e also denote the element in $\mathcal{E}_{\pi_b(\zeta)}$ that's canonically identified to $e \in (\pi^*\mathcal{E})_\zeta$. Then we set

$$T^{(\nabla^{\mathcal{E}},\Phi)}(\zeta_{\text{exit}}) = \lim_{t\to\infty} u(t)$$

making a similar canonical identification. We point out that this limit exists by Lemma 2.5 and that "T" stands for "transport equation."

Definition 2.11. *Over the set of all $(\nabla^{\mathcal{E}}, \Phi)$ that satisfy the decay conditions in Lemma 2.5 above, the operator*

$$(\nabla^{\mathcal{E}}, \Phi) \longmapsto T^{(\nabla^{\mathcal{E}},\Phi)}$$

*is called the **non-abelian X-ray transform**.*

For example, another way to formulate Corollary 2.8 above is that for any g and $\nabla^{\mathcal{E}}$ as stated there, the non-abelian X-ray transform is injective over the set of all Higgs field satisfying the decay condition described therein.

2.5 Prior Research Discussion and Applications

A standard approach for studying injectivity properties of X-ray transforms is via energy identities that was first introduced in [31]. The type of energy estimate that's used in this approach is called the **Pestov identity** (or **Muhometov-Pestov identity**) which over the years has taken many forms as authors apply them in various contexts - see for instance [10], [36], [39], and [40]. The mentioned paper [10] furthermore explains the connection between X-ray transform over connections and inverse problems related to the wave equation. Of recent works, in dimension two the authors of [35] used a Pestov identity to prove solenoidal injectivity of the X-ray transform over tensors, and in their earlier work [34] they proved a "Pestov type identity" to study the attenuated ray transform with a connection and Higgs field.

The paper [37] proceeded to generalize these methods to manifolds of dimensions greater than two, but it didn't cover the case of connections. In [14] the authors generalized the setup in [37] where they studied the X-ray transform for connections and Higgs fields together. For instance, Theorem 2.6 above was proved in [14] in the case of when (M, g) is a compact Riemannian manifold, has strictly convex boundary, has negative sectional curvature, the boundary condition in (2.3) is changed to $u(\gamma(a))$ where $\gamma : [a, b] \to M$ is a unit-speed geodesic traveling between boundary points, and (2.4) is changed to "recording" $u(\gamma(b))$. In this paper we also generalize the Pestov identity proved in [14] to AH spaces, of which a similar formula also appears in [41].

We mention the early work [1] that studied the injectivity of the X-ray transform for one-forms. The work of [43] studied injectivity on tensor fields of rank $m \leq 2$ for analytic simple metrics and a generic class of two-dimensional simple metrics, and proved a stability estimate for the normal operator. Later [42] proved injectivity on two-tensors for all two-dimensional simple metrics which was then extended to tensors of all rank in [35]. The papers [48] and [46] proved injectivity for functions and two-tensors respectively on Riemannian manifolds that admit convex foliations. The paper [13] proved injectivity on tensors of all rank over Riemannian manifolds with negative curvature and strictly convex boundary. We mention that the work [4] characterized the range of the non-abelian X-ray transform on simple surfaces in terms of boundary quantities and that [3] and [29] proved stability estimates for it over Higgs fields. Microlocal techniques have also been applied to the study of the X-ray transform in the presence of conjugate points - we refer the reader to the works [19], [30], [44], and [45].

In the noncompact realm, injectivity for the scalar X-ray transform over hyperbolic spaces was proved in [17], and inversion formulas are given in [2] and [16]. In [23] the authors proved analogous injectivity over Cartan-Hadamard manifolds and in [24] their results were extended to higher dimensions and tensor fields. The paper [33] proved a gauge equivalence for the X-ray transform for connections on Euclidean space assuming a bound on the size of the connection in dimension two.

AH manifolds have gained interest in the past two decades partly due to their role in physics such as the AdS/CFT conjecture made in [25]. The work [6] for instance describes the role of integral geometry in the AdS/CFT correspondence. In this setting, the paper [12] proved injectivity of the X-ray transform for tensor of all orders on asymptotically hyperbolic spaces. On simple AH manifolds, the work [8] generalized their result for the scalar X-ray transform by proving a stability estimate for the normal operator. Analogous to the local problem studied in [48], [9] proved a local injectivity result for the scalar X-ray transform on AH spaces.

Regarding applications of the non-abelian X-ray transform, we also mention its appearance in the theory of solitons when studying the Bogomolny equations in dimensions $2+1$ - see [26] and [49] for details. The paper [20] describes its applications to coherent quantum tomography. For a survey of the non-Abelian X-ray transform and to read more about its applications, we refer the reader to [32].

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3 Well-Definedness of the Non-Abelian X-Ray Transform

In this section we prove Lemma 2.5. Take any complete geodesic $\gamma : (-\infty, \infty) \to M$ and any $e \in \mathcal{E}_{x_0}$ where x_0 is the limit of $\gamma(t)$ in $\overline{M}$ as $t \to -\infty$. Our plan is to

1. prove the existence and uniqueness of the solution to the initial value problem (2.3), on an interval of the form $(-\infty, t_0]$ for some $t_0 \in \mathbb{R}$,
2. argue the existence and uniqueness on the rest of the interval $[t_0, \infty)$ (and hence everywhere),
3. and finally prove that the limit (2.4) exists.

We begin with 1), which we prove by mapping the infinite interval to a bounded one and then applying standard existence and uniqueness results of ordinary differential equations (ODEs). Let $(\rho, y^1, \dots, y^n)$ be asymptotic boundary normal coordinates of $\overline{M}$ containing x_0 in its domain and let (b_j) be a frame for $\mathcal{E}$ over the same domain. Let ${}^{\mathcal{E}}\Gamma^k_{ij}$ denote the connection symbols of $\nabla^{\mathcal{E}}$ with respect to these coordinates and frame (for $\mathcal{E}$). Let $t_0 \in \mathbb{R}$ be a time such that the image of γ is contained in these coordinates for all times $t \in (-\infty, t_0]$. Then, writing $u = u^k b_k$, in these coordinates for $t \in (-\infty, t_0]$ we have that (2.3) becomes the following system of ODEs

$$\frac{du^k}{dt} + {}^{\mathcal{E}}\Gamma^k_{ij}\dot{\gamma}^i u^j + \Phi^k_i u^i = 0, \qquad \lim_{t\to-\infty} u^k = e^k, \qquad k \in \{1, \dots, d\}, \tag{3.1}$$

Let's look at the growth rate of the $\dot{\gamma}^i$'s. By definition, $g = \overline{g}/\rho^2$ for some smooth metric $\overline{g}$ on $\overline{M}$. Since γ has a constant speed one, we have that

$$\overline{g}_{ij}\dot{\gamma}^i\dot{\gamma}^j = \rho^2.$$

Clearly the image $\gamma(-\infty, t_0]$ is a compact subset of our coordinates' domain, and so the matrix in the bilinear form $v \mapsto \overline{g}_{ij}v^i v^j$ has a minimum positive eigenvalue along this set. Hence from the above we get that there exists a $C > 0$ such that each $|\dot{\gamma}^i| \leq C\rho$.

Now, take the diffeomorphism $h : (-\pi/2, s_0] \to (-\infty, t_0]$ given by $h(s) = \tan s$. Making the change of variables $t = h(s)$ in (3.1) gives for each $k \in \{1, \dots, d\}$

$$\frac{du^k}{ds} + {}^{\mathcal{E}}\Gamma^k_{ij}\dot{\gamma}^i\frac{dh}{ds}u^j + \Phi^k_i\frac{dh}{ds}u^i = 0 \quad \text{on } s \in (-\pi/2, 0), \qquad u^k(-\pi/2) = e^k.$$

In other words, the existence and uniqueness of a continuous solution u to these ODEs will prove 1). This in turn will follow from standard results on linear ODEs (see for instance [5]) if we show that the above coefficients ${}^{\mathcal{E}}\Gamma^k_{ij}\dot{\gamma}^i h'$ and $\Phi^k_i h'$ extend continuously to $s = -\pi/2$.

It follows from Lemma 2.3 in [12] (specifically (2.11) there) that there exists a constant $C' > 0$ such that for $t \in (-\infty, t_0]$,

$$\rho \circ \gamma(t) < C' e^t. \tag{3.2}$$

Since by assumption the ${}^{\mathcal{E}}\Gamma_{ij}^k$ and Φ_i^k are in $\rho^\infty C^\infty(\overline{M};\mathbb{C})$, we have that there exists a constant $C'' > 0$ such that for $s \in (-\pi/2, s_0]$ both $\left|{}^{\mathcal{E}}\Gamma_{ij}^k \dot{\gamma}^i h'\right|$ and $\left|\Phi_i^k h'\right|$ are bounded above by

$$C''(\rho \circ \gamma(s))h'(s) \leq C'' C' e^{\tan(s)} \sec^2(s) \to 0 \qquad \text{as } s \to -\pi/2^+$$

Hence indeed ${}^{\mathcal{E}}\Gamma_{ij}^k \dot{\gamma}^i h'$ and $\Phi_i^k h'$ extend continuously to $s = -\pi/2$.

Item 2) follows by applying standard existence and uniqueness theory of ODEs in coordinates and frames of $\mathcal{E}$ as one travels along the geodesic. Item 3) is proved similarly to 1) except one uses Lemma 2.3 in [12] in forward time (for instance, the e^t in (3.2) will change to e^{-t}).

■

4 Geometric Preliminaries

4.1 The b and 0 (Co)Tangent Bundles

Lowering and raising an indices with respect to g provides a bundle isomorphism between the tangent and cotangent bundles over the interior:

$$\flat : TM \longrightarrow T^*M, \qquad \sharp : T^*M \longrightarrow TM.$$

We introduce two more smooth bundles over $\overline{M}$ of rank $n+1$ that will be of use to us. The first is the ***b*-tangent bundle** "${}^bT\overline{M}$," which comes with a canonical smooth map $F : {}^bT\overline{M} \to T\overline{M}$ that has the following two properties:

1. F is a bijection between smooth sections of ${}^bT\overline{M}$ and smooth sections of $T\overline{M}$ that are tangent to the boundary $\partial\overline{M}$.

2. For any fixed point $x \in \overline{M}$, F restricts to a linear homomorphism $F_x : {}^bT_x\overline{M} \to T_x\overline{M}$ that is also an isomorphism when x is in the interior M.

The second is the **0-tangent bundle** "${}^0T\overline{M}$," which is defined similarly as coming with a smooth map $H : {}^0T\overline{M} \to T\overline{M}$ that has the following two properties:

1. H is a bijection between smooth sections of ${}^0T\overline{M}$ and smooth sections of $T\overline{M}$ that vanish at the boundary $\partial\overline{M}$.

2. For any fixed point $x \in \overline{M}$, H restricts to a linear homomorphism $H_x : {}^0T_x\overline{M} \to T_x\overline{M}$ that is also an isomorphism when x is in the interior M.

Of more importance to us will be the dual bundles ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$, which are called the ***b* and 0 cotangent bundles** respectively. They naturally generate pullback maps $F^* : T^*\overline{M} \to {}^bT^*\overline{M}$ and $H^* : T^*\overline{M} \to {}^0T^*\overline{M}$ between the dual bundles respectively that are also bundle isomorphisms over the interior M.

Remark 4.1. Considering that $\flat, \sharp, F, H, F^*, H^*$ are all isomorphisms over the interior M, we will often identify two points in TM, T^*M, ${}^bT\overline{M}\big|_M$, ${}^bT^*\overline{M}\big|_M$, ${}^0T\overline{M}\big|_M$, and ${}^0T^*\overline{M}\big|_M$ as being the same if it's possible to go from one to the other by a composition of the "canonical identification" maps mentioned above.

We mention important bases for the b and 0 cotangent bundles near the boundary. Suppose $(\rho, y^1, \ldots, y^n)$ are boundary coordinates of $\overline{M}$. Then it turns out that

$$F^*\left(\frac{d\rho}{\rho}\right), F^*(dy^1), \ldots, F^*(dy^n)$$

$$H^*\left(\frac{d\rho}{\rho}\right), H^*\left(\frac{dy^1}{\rho}\right), \ldots, H^*\left(\frac{dy^n}{\rho}\right)$$

extend smoothly to the boundary ∂M to be frames of ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$. It's standard to abuse notation by simply writing that ${}^{d\rho}/_{\rho}, dy^1, \ldots, dy^n$ and ${}^{d\rho}/_{\rho}, {}^{dy^1}/_{\rho}, \ldots, {}^{dy^n}/_{\rho}$ are frames for ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$ respectively.

4.2 The b and 0 Cosphere Bundles

Suppose $(\rho, y^1, \ldots, y^n)$ are asymptotic boundary normal coordinates of $\overline{M}$ as described in Section 2.2 above. We know that T^*M has a fiber metric g. Thus the maps $F^* : T^*\overline{M} \to {}^bT^*\overline{M}$ and $H^* : T^*\overline{M} \to {}^0T^*\overline{M}$ push g to become fiber metrics on ${}^bT^*\overline{M}\big|_M$ and ${}^0T^*\overline{M}\big|_M$, which we denote by g_b and g_0 respectively. If we consider the boundary frames for ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$ introduced in Section 4.1 above, we have that these metrics are given by (c.f. (2.2))

$$\begin{aligned} \left|\eta_0 \tfrac{d\rho}{\rho} + \eta_\mu dy^\mu\right|^2_{g_b} &= \eta_0^2 + \rho^2 h^{\mu\nu}\eta_\mu\eta_\nu, \\ \left|\overline{\eta}_0 \tfrac{d\rho}{\rho} + \overline{\eta}_\mu \tfrac{dy^\mu}{\rho}\right|^2_{g_0} &= \overline{\eta}_0^2 + h^{\mu\nu}\overline{\eta}_\mu\overline{\eta}_\nu, \end{aligned} \tag{4.2}$$

where $(h^{\mu\nu})$ denotes the inverse matrix of $(h_{\mu\nu})$. From here we see that both g_b and g_0 extend smoothly to all of ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$ respectively. This allows us to define the **unit cosphere bundles** in ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$:

$${}^bS^*\overline{M} = \left\{\zeta \in {}^bT^*\overline{M} : |\zeta|_{{}^bg} = 1\right\},$$

$${}^0S^*\overline{M} = \left\{\overline{\zeta} \in {}^bT^*\overline{M} : |\overline{\zeta}|_{{}^0g} = 1\right\}.$$

We note that in [12], they use the notation "$\overline{S^*M}$" instead of "${}^bS^*\overline{M}$." We let $\pi_b : {}^bS^*\overline{M} \to \overline{M}$ and $\pi_0 : {}^0S^*\overline{M} \to \overline{M}$ denote the natural projection maps.

Remark 4.3. Similarly to the remark made in Remark 4.1, we will often identify two points in SM, ${}^bS^*\overline{M}$, and ${}^0S^*\overline{M}$ to be the same if it's possible to go from one to the other by a composition of the maps mentioned there.

As a last note, we point out that it's easy to see that both ${}^bS^*\overline{M}$ and ${}^0S^*\overline{M}$ are smooth embedded submanifolds with boundary of ${}^bT^*\overline{M}$ and ${}^0T^*\overline{M}$. We also note that by (4.2), g_b degenerates over $\partial\overline{M}$ (i.e. stops being positive definite) while g_0 does not. In particular this implies that ${}^bS^*\overline{M}$ is not compact while ${}^0S^*\overline{M}$ is compact.

4.3 Splitting the Tangent Bundle

Next we define a natural Riemannian metric on the tangent space TM, called the **Sasaki metric**, generated by g. We recommend that when checking many of the claims below, to check them above the center of normal coordinates since in many cases the expressions simplify considerably due to the vanishing of the Christoffel symbols and the first order partials of g. Consider the tangent bundle's projection map $\pi : TM \to M$ and its differential $d\pi : TTM \to TM$. There is another natural map between the corresponding tangent spaces called the **connection map**: $\mathcal{K} : TTM \to TM$, which is defined as follows. Take any $\omega \in TTM$ and let $\alpha : (a, b) \to M$ be a smooth curve and $V : (a, b) \to TM$ a smooth vector field along α such that $(\alpha, V)'(0) = \omega$. Then we set $\mathcal{K}(\omega)$ to be the covariant derivative

$$\mathcal{K}(\omega) := \frac{DV}{dt}(0).$$

To check that this is independent of the α and V that we choose, a quick computation shows that taking coordinates (x^i) of M and the coordinates $v^i\partial/\partial x^i \mapsto (x^i, v^i)$ of TM, $\mathcal{K}$ is given by

$$\mathcal{K}\left(\alpha^i \frac{\partial}{\partial x^i}\bigg|_{v^i\partial/\partial x^i} + \beta^i \frac{\partial}{\partial v^i}\bigg|_{v^i\partial/\partial x^i}\right) = \left(\beta^k + \Gamma^k_{ij}\alpha^i v^j\right)\frac{\partial}{\partial x^k},$$

where Γ^k_{ij} are the Christoffel symbols of g. Next, an easy exercise shows that the kernels of these two maps partition the tangent bundle's tangent space at any $v \in T_xM$:

$$T_vTM = \widetilde{\mathcal{H}}_v \oplus \widetilde{\mathcal{V}}_v \tag{4.4}$$

where

$$\widetilde{\mathcal{H}}_v = \ker \mathcal{K}|_{T_vTM} \qquad \text{and} \qquad \widetilde{\mathcal{V}}_v = \ker d\pi|_{T_vTM}.$$

The "$\widetilde{\mathcal{V}}$" stands for "vertical" because it can be imagined as being a tangent subspace at v standing vertically above x, while the "$\widetilde{\mathcal{H}}$" stands for "horizontal." As one can check, both spaces are canonically identified (i.e. isomorphically mapped to) with T_xM by the restricted maps

$$d\pi : \widetilde{\mathcal{H}}_v \longrightarrow T_xM,$$
$$\mathcal{K} : \widetilde{\mathcal{V}}_v \longrightarrow T_xM.$$

With this splitting in hand, the Sasaki metric G on TM is defined as follows: for any $\omega, \varsigma \in T_vTM$,

$$\langle \omega, \varsigma\rangle_G = \langle d\pi(\omega), d\pi(\varsigma)\rangle_g + \langle \mathcal{K}(\omega), \mathcal{K}(\varsigma)\rangle_g.$$

It follows immediately that (4.4) is an orthogonal decomposition with respect to G.

We will only work with unit speed geodesics and hence most of our work will be done on the unit sphere bundle

$$SM = \Big\{ v \in TM : |v|_g = 1 \Big\}.$$

With a choice of unit normal, the Sasaki metric on TM induces a metric on SM, which we'll also call the Sasaki metric and denote by G, relying on context to differentiate the two. It's not hard to see that at any $v \in SM$ the tangent space of SM splits into the form

$$T_v SM = \widetilde{\mathcal{H}}_v \oplus \mathcal{V}_v$$

where $\mathcal{V}_v$ is the subspace of $\widetilde{\mathcal{V}}_v$ that's G-perpendicular to the unit normals to the "sphere" $S_x M$ above x. Now, let X denote the geodesic vector field over SM. It's easy to check that X always lies in $\mathcal{H}_v$ for all $v \in SM$ and hence we obtain the standard splitting

$$T_v SM = (\mathbb{R} X_v) \oplus \mathcal{H}_v \oplus \mathcal{V}_v \tag{4.5}$$

where $\mathcal{H}_v$ denotes the orthogonal complement of $\mathbb{R}X_v$ in $\widetilde{\mathcal{H}}_v$. For future use, we point out that restrictions of $d\pi$ and $\mathcal{K}$ map

$$\begin{aligned} d\pi : \mathcal{H}_v &\longrightarrow \big\{ v^\perp \big\} \subseteq T_x M, \\ \mathcal{K} : \mathcal{V}_v &\longrightarrow \big\{ v^\perp \big\} \subseteq T_x M. \end{aligned} \tag{4.6}$$

4.4 Integration on the (Co)Sphere Bundle

Since SM has a Riemannian metric G, it has a Riemannian density and hence the Lebesgue measure generated by it (latter two are independent of orientation). Hence we may perform Lebesgue integration on SM with respect to G. If (x^i) are local coordinates of M and we take the coordinates $v^i \partial/\partial x^i \mapsto (x^i, v^i)$ of TM, then it turns out that the integral of any function $f \in L^1(SM)$ supported over our coordinates is given by the iterated integral

$$\int f = \int f\big(x^1, \dots, x^n, v^1, \dots, v^n\big) dS_x\big(v^1, \dots, v^n\big) \sqrt{\det g} dx^1 \dots dx^n,$$

where $(v^1, \dots, v^n)$ are on the sphere $g_{ij} v^i v^j = 1$ and dS_x is the Lebesgue measure on $S_x M$ induced by $T_x M$ with inner product g_x. We refer the reader to Section 3.6.2 in [38] for a proof. We point out that the (total) measure of $S_x M$ is the Euclidean surface area of the Euclidean n-sphere for all $x \in M$.

As an example of the usefulness of this observation, we point out that since $\sqrt{\det g}$ is $\rho^{-(n+1)}$ times "something smooth" on $\overline{M}$ we immediately get that any function of the form $\rho^{n+1} L^\infty(SM)$ is integrable.

4.5 Splitting the Connection Over the Unit Tangent Bundle

Let us take the natural projection map $\pi : SM \to M$. The pullback (vector) bundle $\pi^* \mathcal{E}$ over SM is defined as the set obtained by taking any point $x \in M$ and attaching a copy of $\mathcal{E}_x$ to every point of the sphere $S_x M$ above it. Formally,

$$\pi^* \mathcal{E} := \big\{ (v, e) : v \in SM, e \in \mathcal{E}_{\pi(v)} \big\}.$$

We often canonically identify (v,e) with $e \in \mathcal{E}_{\pi(v)}$ when v is fixed. To every fiber $(\pi^*\mathcal{E})_v$ we impose the inner product space structure of $(\mathcal{E}_x, \langle\cdot,\cdot\rangle_{\mathcal{E}_x})$. If (b_i) is a smooth frame for $\mathcal{E}$, then we turn $\pi^*\mathcal{E}$ into a smooth vector bundle over SM (with smooth inner product) by declaring frames for $\pi^*\mathcal{E}$ of the form[3] $(\pi^* b_i)$ to be smooth. The pullback connection $\nabla^{\pi^*\mathcal{E}} = \pi^*\nabla^{\mathcal{E}}$ in $\pi^*\mathcal{E}$ is defined to be the unique connection so that if $\Omega : TM \to TTM$ and $u : M \to \mathcal{E}$ are smooth, then

$$\nabla^{\pi^*\mathcal{E}}_{\Omega}(\pi^* u) = \pi^*\big(\nabla^{\mathcal{E}}_{d\pi(\Omega)} u\big).$$

Having defined the splitting of the unit tangent bundle in (4.5), we now define a natural splitting of the connection of any section $u : SM \to \pi^*\mathcal{E}$ in the following form:

$$\nabla^{\pi^*\mathcal{E}} u = \overset{X}{\nabla}{}^{\pi^*\mathcal{E}} u + \overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u + \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} u. \tag{4.7}$$

Let's start by defining $\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u$. We have that the full connection $\nabla^{\pi^*\mathcal{E}} u$ is a tensor of the form $C^\infty(SM; T^*SM \otimes \pi^*\mathcal{E})$. Now, consider the same tensor but with the first index raised with respect to G: $\big[\nabla^{\pi^*\mathcal{E}} u\big]^\sharp \in C^\infty(SM; TSM \otimes \pi^*\mathcal{E})$. Next, it's an easy exercise to check that there exists a unique map

$$P_{\mathcal{H}} : C^\infty(SM; TSM \otimes \pi^*\mathcal{E}) \longrightarrow C^\infty(SM; TSM \otimes \pi^*\mathcal{E}) \tag{4.8}$$

that satisfies

$$P_{\mathcal{H}}(\omega \otimes e) = (\mathrm{proj}_{\mathcal{H}}\omega) \otimes e$$

where $\mathrm{proj}_{\mathcal{H}} : T_v TM \to \mathcal{H} \subseteq T_v TM$ is the orthogonal projection map onto $\mathcal{H}$. We then define

$$\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u := P_{\mathcal{H}}\Big(\big[\nabla^{\pi^*\mathcal{E}} u\big]^\sharp\Big).$$

We define $\overset{X}{\nabla}{}^{\pi^*\mathcal{E}} u$ and $\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} u$ the same way but instead use analogous map $P_{\mathbb{R}X}$, $\mathrm{proj}_{\mathbb{R}X}$ and $P_{\mathcal{V}}$, $\mathrm{proj}_{\mathcal{V}}$ respectively. However instead of using $\overset{X}{\nabla}{}^{\pi^*\mathcal{E}} u$, it's more common to use the related quantity

$$\mathbb{X} u := \nabla^{\pi^*\mathcal{E}}_X u. \tag{4.9}$$

We note that one can be computed from the other and so we often record the decomposition (4.7) instead as

$$\nabla^{\pi^*\mathcal{E}} u = \Big(\mathbb{X} u, \overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} u\Big). \tag{4.10}$$

The second two components are called the **horizontal and vertical derivatives** of u respectively. It's convenient to change the interpretation of the latter two derivatives as follows.

We define the bundle N over SM obtained by attaching to every $v \in S_x M$ a copy of $\{v^\perp\} \subseteq T_x M$. Formally,

$$N = \big\{(v,w) : v \in S_x M \ \text{ where } \ x \in M \ \text{ and } \ w \in \{v^\perp\}\big\}.$$

[3]Applying π^* to b_i means "$b_i \circ \pi$."

To every fiber N_v we impose the inner product space structure of $\left(\{v^\perp\}, g_x\right)$ which we denote by "$\langle\cdot,\cdot\rangle_{N_v}$." It's an easy exercise to show that this is a smooth subbundle of π^*TM. By (4.6) we can think of $d\pi$ and $\mathcal{K}$ as mapping

$$\begin{aligned} d\pi : \mathcal{H}_v &\longrightarrow N_v, \\ \mathcal{K} : \mathcal{V}_v &\longrightarrow N_v. \end{aligned} \tag{4.11}$$

Above every $v \in TM$, the maps $P_{\mathcal{H}}$ and $P_{\mathcal{V}}$ above map into $\mathcal{H}_v \otimes (\pi^*\mathcal{E})_v$ and $\mathcal{V}_v \otimes (\pi^*\mathcal{E})_v$ respectively. Hence using the identification (4.11) we can think of the horizontal and vertical derivatives as both being $N \otimes \mathcal{E}$-valued:

$$\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u \in C^\infty(SM; N \otimes \pi^*\mathcal{E}) \qquad \text{and} \qquad \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} u \in C^\infty(SM; N \otimes \pi^*\mathcal{E}).$$

The reason this is useful is that it becomes natural to apply well-known adjoint formulas for the horizontal and vertical derivatives over this space. In particular, it turns out that there are differential operators

$$\begin{aligned} \overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}} &: C^\infty(SM; N \otimes \pi^*\mathcal{E}) \longrightarrow C^\infty(SM; \pi^*\mathcal{E}), \\ \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}} &: C^\infty(SM; N \otimes \pi^*\mathcal{E}) \longrightarrow C^\infty(SM; \pi^*\mathcal{E}), \end{aligned} \tag{4.12}$$

with the property that if $u \in C^\infty(SM; \pi^*\mathcal{E})$ and $v \in C^\infty(SM; N \otimes \pi^*\mathcal{E})$ are such that at least one of them is of compact support, then

$$\langle \overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u, v\rangle_{L^2(N\otimes\pi^*\mathcal{E})} = -\langle u, \overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}} v\rangle_{L^2(\pi^*\mathcal{E})} \;\text{ and }\; \langle \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} u, v\rangle_{L^2(N\otimes\pi^*\mathcal{E})} = -\langle u, \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}} v\rangle_{L^2(\pi^*\mathcal{E})}.$$

We discuss this in more detail in Section 5.1 below. The operators in (4.12) are naturally called the **horizontal and vertical divergences** respectively.

For future reference, we end this subsection with two more definitions. First, we define the differential operator

$$\mathbb{X} : C^\infty(SM; N \otimes \pi^*\mathcal{E}) \to C^\infty(SM; N \otimes \pi^*\mathcal{E}),$$

differentiated from the $\mathbb{X}$ introduced in (4.9) by context, to be the unique operator satisfying that for any $Z \otimes b \in C^\infty(SM; N \otimes \pi^*\mathcal{E})$,

$$\mathbb{X}(Z \otimes b) = X(Z) \otimes b + Z \otimes \mathbb{X}(b), \tag{4.13}$$

where $X(Z)$ at any point $v \in TM$ denotes (utilizing proper identifications) the covariant derivative of Z along the geodesic γ with initial velocity v at time $t = 0$:

$$X(Z)\big|_v = \frac{D_\gamma Z}{dt}(0).$$

We point out that $v = d\pi_v(X)$, which motivates the notation "$X(Z)$." Since $Z \perp \dot\gamma$ implies that ${}^{D_\gamma Z}/_{dt} \perp \dot\gamma$, we see that $\mathbb{X}$ indeed maps into smooth sections of $N \otimes \pi^*\mathcal{E}$ (i.e. not simply into $\pi^*TM \otimes \pi^*\mathcal{E}$).

Second, we define the analogous bundle ${}^0N$ over ${}^0S\overline{M}$ given by

$${}^0N = \left\{ \left(\overline{\zeta}, \vartheta\right) : \overline{\zeta} \in {}^0S_x^*\overline{M} \;\text{ where }\; x \in \overline{M} \;\text{ and }\; \vartheta \in \left\{\overline{\zeta}^\perp\right\} \subseteq {}^0T_x\overline{M} \right\},$$

where $\{\vartheta^\perp\}$ is computed using g_0. We impose to every fiber ${}^0N_\xi$ the vector space structure of $(\{\xi^\perp\}, (g_0)_\xi)$. This is also a smooth subbundle of $\pi_0^{*0}T^*\overline{M}$.

Remark 4.14. Similarly to Remark 4.1, we will often identify two points in the pullback bundles π^*TM, $\pi_b^{*\,b}T^*\overline{M}\big|_M$, and $\pi_0^{*\,0}T^*\overline{M}\big|_M$ to be the same if it's possible to go from one to the other by first moving to the associated bundle fiber over M via identification, then by compositions of the maps described in that remark, and then lifting to the associated bundle fiber on the (co)sphere. For example, we often identify $(v, w) \in \pi^*TM$ to be the same as $\big(H^*\big(v^\flat\ \big), H^*\big(w^\flat\ \big)\big)$. It's not hard to see that this identification restricts to $N \to {}^0N$.

4.6 Curvatures

We now cover the curvature operators of $\mathcal{E}$, $\pi^*\mathcal{E}$, and simply the metric g. We start with the first one. The operator $\nabla^{\mathcal{E}}$ maps between the following spaces of sections:

$$\nabla^{\mathcal{E}} : C^\infty\big(\overline{M}; \mathcal{E}\big) \longrightarrow C^\infty\big(\overline{M}; T^*\overline{M} \otimes \mathcal{E}\big)$$

Let $\Lambda^k\big(T^*\overline{M}\big)$ denote the bundle of covariant alternating k-tensors and let

$$C^\infty\big(\overline{M}; \Lambda^k\big(T^*\overline{M}\big) \otimes \mathcal{E}\big)$$

denote the space of smooth sections of $T^*M \otimes \ldots \otimes T^*M \otimes \mathcal{E}$ that are alternating in their first k arguments. The operators

$$\nabla^{\mathcal{E}} : C^\infty\big(\overline{M}; \Lambda^k(T^*M) \otimes \mathcal{E}\big) \longrightarrow C^\infty\big(\overline{M}; \Lambda^{k+1}\big(T^*\overline{M}\big) \otimes \mathcal{E}\big)$$

are defined to be the unique operators that satisfy that for any $\theta \in C^\infty\big(\overline{M}; \Lambda^k\big(T^*\overline{M}\big)\big)$ and any $u \in C^\infty(M; \mathcal{E})$

$$\nabla^{\mathcal{E}}(\theta \otimes u) = d\theta \otimes u + (-1)^k \theta \wedge \nabla^{\mathcal{E}} u,$$

where $\theta \wedge \nabla^{\mathcal{E}} u$ denotes the wedge-like product:

$$\theta \wedge \nabla^{\mathcal{E}} u(v_1, \ldots, v_{k+1}, l) = \frac{1}{k!1!} \sum_{\sigma \in S_{k+1}} \theta\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big) \nabla^{\mathcal{E}} u\big(v_{\sigma(k+1)}, l\big),$$

where S_{k+1} denotes the set of permutations of $k+1$ elements.

The **curvature** of $\nabla^{\mathcal{E}}$ is defined to be

$$f^{\mathcal{E}} := \nabla^{\mathcal{E}} \circ \nabla^{\mathcal{E}} : C^\infty\big(\overline{M}; \mathcal{E}\big) \longrightarrow C^\infty\big(\overline{M}; \Lambda^2\big(T^*\overline{M}\big) \otimes \mathcal{E}\big).$$

A straightforward computation shows that in any coordinates (x^i) of $\overline{M}$ and any frame (b_i) of $\mathcal{E}$, the curvature $f^{\mathcal{E}}$ applied to any smooth section $u \in C^\infty\big(\overline{M}; \mathcal{E}\big)$ is given by

$$f^{\mathcal{E}} u = u^l \left(\frac{\partial^{\mathcal{E}} \Gamma^k_{jl}}{\partial x^i} - {}^{\mathcal{E}}\Gamma^m_{il} {}^{\mathcal{E}}\Gamma^k_{jm} - \frac{\partial^{\mathcal{E}} \Gamma^k_{il}}{\partial x^j} + {}^{\mathcal{E}}\Gamma^m_{jl} {}^{\mathcal{E}}\Gamma^k_{im} \right) dx^i \otimes dx^j \otimes b_k. \tag{4.15}$$

where ${}^{\mathcal{E}}\Gamma^k_{ij}$ are the connection symbols of $\nabla^{\mathcal{E}}$ with respect to these coordinates and frames. The resemblance of this tensor to the Riemann curvature tensor is the motivation for the name of $f^{\mathcal{E}}$.

Next we define a curvature operator associated to $f^{\mathcal{E}}$ which acts over SM. Notice that $f^{\mathcal{E}}$ can be viewed as a $C^\infty\big(\overline{M};\Lambda^2\big(T^*\overline{M}\big)\otimes\mathcal{E}\otimes\mathcal{E}^*\big)$ tensor field where u is placed in the fourth argument in (4.15) (i.e. the argument of $\mathcal{E}^*$). Hence it can also be canonically identified with a smooth map, denoted by the same letter, of the form

$$f^{\mathcal{E}}:C^\infty\big(\overline{M};T\overline{M}\big)\times C^\infty\big(\overline{M};\mathcal{E}\big)\longrightarrow C^\infty\big(\overline{M};T^*\overline{M}\otimes\mathcal{E}\big).$$

In our coordinates and frames it is given by the following: if ${f_{ij}}^k{}_l$ denotes the tensor component written out in the "$(\ldots)$" in (4.15), then for any $x\in\overline{M}$, $v\in T_x\overline{M}$, and $e\in\mathcal{E}_x$,

$$f^{\mathcal{E}}_x(v,e)=e^l{f_{ij}}^k{}_l v^i dx^j\otimes b_k. \tag{4.16}$$

We define the **curvature operator** associated to $f^{\mathcal{E}}$ to be the map

$$F^{\mathcal{E}}:C^\infty(SM;\pi^*\mathcal{E})\longrightarrow C^\infty(SM;N\otimes\pi^*\mathcal{E})$$

given by the following. For any $x\in M$, $v\in S_xM$, and $e\in(\pi^*\mathcal{E})_v$,

$$F^{\mathcal{E}}_v(e):=P_{N\otimes\pi^*\mathcal{E}}\Big(\big[f^{\mathcal{E}}_x(v,e)\big]^\sharp\Big). \tag{4.17}$$

where $\sharp$ raises the first index of $f^{\mathcal{E}}_x(v,e)$ and $P_{N\otimes\pi^*\mathcal{E}}:TM\otimes\pi^*\mathcal{E}\to N\otimes\pi^*\mathcal{E}$ denotes projecting the first component of a tensor product perpendicularly onto the normal bundle (i.e. TM onto N - c.f. (4.8)).

The last curvature quantity that we want to establish notation for is the ordinary curvature of g. Let

$$R:C^\infty(M;TM)\times C^\infty(M;TM)\times C^\infty(M;TM)\longrightarrow C^\infty(M;TM)$$

denote the Riemann curvature endomorphism given by

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

where ∇ is the Levi-Civita connection and $[\cdot,\cdot]$ is the Lie bracket. Recall that R is multilinear over $C^\infty(M)$ and thus has well-defined restrictions to $(T_xM)^3\to T_xM$ for any fixed $x\in M$. We define operators, denoted by the same letter,

$$R:C^\infty(SM;N)\longrightarrow C^\infty(SM;N)$$

$$R:C^\infty(SM;N\otimes\pi^*\mathcal{E})\longrightarrow C^\infty(SM;N\otimes\pi^*\mathcal{E})$$

given by the following, for any $x\in M$, $v\in S_xM$, $w\in N_v$, and $e\in(\pi^*\mathcal{E})_v$,

$$R_v(w):=R_x(w,v)v \qquad \text{and} \qquad R_v(w\otimes e):=[R_x(w,v)v]\otimes e.$$

5 Gauge Equivalence of Connections and Higgs fields

In this section we build the necessary tools to prove Theorem 2.6, and then prove it at the end.

5.1 Pestov Identity

In this section we prove the following version of the **Pestov Identity** with a connection on asymptotically hyperbolic (AH) spaces.

Theorem 5.1. *Suppose that* $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ *for some* $\alpha > {}^{(n+1)}\!/_2$. *Then*[4]

$$\|\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u\|_{L^2}^2 = \|\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2 - \langle R\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} - \langle F^{\mathcal{E}}u, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} + n\|\mathbb{X}u\|_{L^2}$$

where L^2 *stands for* $L^2(SM; N \otimes \pi^*\mathcal{E})$ *in the first four quantities and* $L^2(SM; \pi^*\mathcal{E})$ *in the last one.*

Intuitively speaking it studies how the "energy" (i.e. L^2-norm squared) changes when one switches the order of $\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}$ and $\mathbb{X}$. The theorem is proved by simply starting with $\|\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u\|_{L^2}^2$ and then applying L^2-adjoint relations and commutator formulas until one arrives at $\|\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2$. The factor ρ^α for $\alpha > {}^{(n+1)}\!/_2$ is included to provide sufficient decay at infinity to make all of the functions in question L^2-integrable. We multiply this factor by $C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ because, as we will show below, all of the differential operators involved extend smoothly to ${}^0S^*\overline{M}$ and hence preserve this decay rate.

The following lemma provides us with the set of adjoint relations mentioned above that we need.

Lemma 5.2. *The following are true, where* $\alpha, \beta \in \mathbb{R}$ *are real numbers and all* L^2 *stand for appropriate* $L^2(SM; \dots)$ *spaces.*

1. *If* $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$, *then* $\mathbb{X}u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ *as well. Furthermore, if* $w \in \rho^\beta C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ *is such that* $\alpha + \beta \geq n + 1$, *then*
$$\langle \mathbb{X}u, w\rangle_{L^2} = -\langle u, \mathbb{X}w\rangle_{L^2}.$$
2. *If* $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$, *then* $\mathbb{X}u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$ *as well. Furthermore, if* $w \in \rho^\beta C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$ *is such that* $\alpha + \beta \geq n + 1$, *then*
$$\langle \mathbb{X}u, w\rangle_{L^2} = -\langle u, \mathbb{X}w\rangle_{L^2}.$$
3. *If* $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ *and* $w \in \rho^\beta C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$, *then* $\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$ *and* $\overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}w \in \rho^\beta C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$. *If furthermore* $\alpha + \beta \geq n + 1$, *then*
$$\langle \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, v\rangle_{L^2} = -\langle u, \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}v\rangle_{L^2}.$$

In other words, the operators $\mathbb{X}$, $\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}$, and $\overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}$ don't affect decay rates at "infinity" (i.e. ∂M) and their well-known adjoint relations still hold on AH spaces. To prove this lemma, we use the following compactly supported version of it:

[4]Here we're implicitly restricting to the interior so that we may apply the differential operators involved and integrate.

Lemma 5.3. *The following are true.*

1. *If $u, w \in C^\infty(SM; \pi^*\mathcal{E})$ are such that at least one of them is of compact support, then*
$$\langle \mathbb{X}u, w\rangle_{L^2} = -\langle u, \mathbb{X}w\rangle_{L^2}.$$
2. *If $u, w \in C^\infty(SM; N \otimes \pi^*\mathcal{E})$ are such that at least one of them is of compact support, then*
$$\langle \mathbb{X}u, w\rangle_{L^2} = -\langle u, \mathbb{X}w\rangle_{L^2}.$$
3. *If $u \in C^\infty(SM; \pi^*\mathcal{E})$ and $w \in C^\infty(SM; N \otimes \pi^*\mathcal{E})$ are such that at least one of them is of compact support, then*
$$\langle \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} u, w\rangle_{L^2} = -\langle u, \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}} w\rangle_{L^2}.$$

As mentioned before, point 3) in Lemma 5.3 here is stated in [14] and points 1) and 2) are implicitly used by the authors in the same paper.[5] So we do not provide a proof of Lemma 5.3.

Proof of Lemma 5.2 part 1):

Let u be as described in part 1). Let (ρ, y^μ) be asymptotic boundary normal coordinates of $\overline{M}$ contained in a collar neighborhood $\mathcal{C}_\varepsilon$ of ∂M as described in Section 2.2. For simplicity of notation, we also write these coordinates as (x^i). Consider the coordinates $v^i \partial/\partial x^i \mapsto (x^i, v^i)$ of TM. Next, let (b_i) be a frame for $\mathcal{E}$ over these coordinates (of $\overline{M}$). We write u componentwise as
$$u = u^i \pi^* b_i \qquad \text{and} \qquad u = {}^0u^i \pi_0^* b_i, \tag{5.4}$$
where the first u here is the element in $C^\infty(SM; \pi^*\mathcal{E})$ canonically identified with $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$. We have that
$$\mathbb{X}u \text{ over } SM = \big[X\big(u^k\big) + {}^{\mathcal{E}}\Gamma^k_{ij} v^i u^j\big]\pi^* b_k.$$
where ${}^{\mathcal{E}}\Gamma^k_{ij}$ denote the connection symbols of $\nabla^{\mathcal{E}}$ with respect to (x^i) and (b_i), and $v \in SM$ denotes the position on SM. We will now pull this quantity back to $\pi_0^*\mathcal{E}$.

Let X_0 denote the pushforward of X under the canonical identification $H^* \circ \flat : TM \to {}^0T^*\overline{M}\big|_M$ (i.e. $X_0 = d(H^* \circ \flat\,)(X)$). Consider the frame $({}^{dx^i}/\rho)$ of ${}^0T^*\overline{M}$ whose points we will typically write as $\overline{\eta}_i {}^{dx^i}/\rho$. It's easy to see that $\pi^* b_k$ and $\pi_0^* b_k$ are canonically identified, and hence $u^k = {}^0u^k$ when evaluated at identified points. Similarly it's easy to see that if $v^i \partial/\partial x^i$ and $\overline{\eta}_i {}^{dx^i}/\rho$ are identified points, then $v^i = \rho^{-1} g^{ii'} \overline{\eta}_{i'}$. Hence
$$\mathbb{X}u \text{ over } {}^0S^*M\big|_M = \Big[X_0\big({}^0u^k\big) + {}^{\mathcal{E}}\Gamma^k_{ij} \rho^{-1} g^{ii'} \overline{\eta}_{i'} {}^0u^j\Big]\pi_0^* b_k. \tag{5.5}$$

Observe that $g^{ii'}$ is ρ^2 times something smooth on $\overline{M}$. Consider the coordinates $\xi_i dx^i \mapsto (x^i, \xi_i)$ of $T^*\overline{M}$. In (2.3) of [12] the authors write out an explicit equation for X over T^*M

[5] Actually, the authors in the mentioned work are working over compact manifolds with boundary and not AH spaces, but that doesn't matter due to the compact support assumptions here.

in asymptotic boundary normal coordinates (recall the convention about Greek and Latin indices):

$$X = \rho^2 \xi_0 \frac{\partial}{\partial \rho} + \rho^2 h^{\mu\nu} \xi_\mu \frac{\partial}{\partial y^\nu} - \left[\rho\big(\xi_0^2 + |\xi|_h^2\big) + \frac{1}{2}\rho^2 \partial_\rho |\xi|_h^2 \right] \frac{\partial}{\partial \xi_0} - \frac{1}{2}\rho^2 \partial_{y^k} |\xi|_h^2 \frac{\partial}{\partial \xi_k}$$

where $|\xi|_h^2 = h^{\mu\nu}\xi_\mu\xi_\nu$. Since $\overline{\eta}_i = \rho\xi_i$, it's a quick calculation to show that

$$X_0 = \rho\overline{\eta}_0 \frac{\partial}{\partial \rho} + \rho h^{\mu\nu} \overline{\eta}_\mu \frac{\partial}{\partial y^\nu} - \left[|\overline{\eta}|_h^2 + \frac{1}{2}\rho \partial_\rho |\overline{\eta}|_h^2 \right] \frac{\partial}{\partial \eta_0} + \left[\overline{\eta}_0 \overline{\eta}_\mu - \frac{1}{2}\rho \partial_{y^\mu} |\overline{\eta}|_h^2 \right] \frac{\partial}{\partial \overline{\eta}_\mu}.$$

We only need the fact that X_0 extends smoothly to the boundary and that the coefficient of $\partial/\partial\rho$ is ρ times something smooth on ${}^0T^*\overline{M}$ (and hence on ${}^0S^*\overline{M}$), because from this we quickly get that (5.5) is indeed in $\rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$.

Now suppose that w is as in 1). We will prove the equality in 1) by multiplying w by a compactly supported (smooth) bump function, use Lemma 5.3 above, and then let the support of the bump function go out to infinity. To construct the suitable family of bump functions, let $f_1 : [0,\infty) \to [0,\infty)$ be a smooth function that is identically zero on $[0, 1/2]$, increasing on $[1/2, 1]$, and then identically one on $[1,\infty)$ (see Lemma 2.21 in [21] for an explicit construction). For any $\delta > 0$, let $f_\delta : [0,\infty) \to [0,\infty)$ denote the function $f_\delta(x) = f(x/\delta)$. Finally, for $\delta < \varepsilon$ let $\phi_\delta : \overline{M} \to [0,\infty)$ denote the one parameter family of bump functions given by

$$\phi_\delta(x) = \begin{cases} f_\delta \circ \rho(x) & \rho(x) < \delta \\ 1 & \text{otherwise} \end{cases}.$$

By Lemma 5.3 we have that

$$\langle \mathbb{X}u, \phi_\delta w\rangle_{L^2} = -\langle u, \mathbb{X}(\phi_\delta w)\rangle_{L^2}$$

since $\phi_\delta w$ is compactly supported. Applying the product rule on the right-hand side gives

$$\langle \mathbb{X}u, \phi_\delta w\rangle_{L^2} = -\langle u, \phi_\delta \mathbb{X}w\rangle_{L^2} - \langle u, X(\phi_\delta)w\rangle_{L^2}. \tag{5.6}$$

We now let $\delta \to 0^+$ and show that this equation tends to the equality in 1). By what we proved above, we have that $\langle \mathbb{X}u, w\rangle_{\pi^*\mathcal{E}}$ and $\langle u, \mathbb{X}w\rangle_{\mathcal{E}}$ are in $\rho^{\alpha+\beta}C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ and hence in $\rho^{n+1}L^\infty(SM; \pi^*\mathcal{E})$. Next, differentiating in δ demonstrates that ϕ_δ monotonely increases to the identically one function as $\delta \to 0^+$. Hence by the dominated convergence theorem, we get that the first two terms in the above equation tend to $\langle \mathbb{X}u, w\rangle_{L^2}$ and $-\langle u, \mathbb{X}w\rangle_{L^2}$ respectively as $\delta \to 0^+$.

Hence we will have proved 1) if we can show that the third term in the above equation tends to zero as $\delta \to 0^+$. This will follow if we show that for any compact set $K \subseteq \overline{M}$ contained in the domain of some interior coordinates (z^i) of $\overline{M}$ or boundary coordinates $(\rho, y^\mu) = (x^i)$ as above, then

$$\int_{\pi^{-1}[K]} \langle u, X(\phi_\delta)w\rangle_{\pi^*\mathcal{E}} \longrightarrow 0 \qquad \text{as} \quad \delta \to 0^+.$$

If K is contained in the domain of interior coordinates, then this follows immediately since $\phi_\delta \equiv 1$ on K for sufficiently small $\delta > 0$. So suppose that K is contained in the domain of our boundary coordinates $(\rho, y^\mu) = (x^i)$. Writing the above integral in these coordinates as in Section 4.4 gives

$$\left|\int_K \langle u, X(\phi_\delta)w\rangle_{\pi^*\mathcal{E}} dS_x(v)\sqrt{\det g}dx\right| \leq \sup\left|\langle u, w\rangle_{\pi^*\mathcal{E}}\sqrt{\det g}\right|\omega_n \int_K |X(\phi_\delta)|dx.$$

where the sup... is finite due to the decay of u and w and ω_n denotes the surface area of the Euclidean n-sphere. The explicit equation for X in coordinates of TM (e.g. see page 104 in [22]) gives that $X(\phi_\delta) = v^0 f'_\delta(\rho)$, which we note is supported in $\{\rho \leq \delta\}$. Since $g = {}^{(d\rho^2 + h_{\mu\nu}dy^\mu dy^\nu)}/_{\rho^2}$ and $|v|_g = 1$, we have that $|v^0| \leq \rho$. Hence the integrand

$$\int_K |X(\phi_\delta)|dx \leq \int_{\mathrm{dom}\,(y^\mu)} \int_0^\varepsilon \delta f'_\delta(\rho)d\rho dy = \delta \int_{\mathrm{dom}\,(y^\mu)} dy \longrightarrow 0 \qquad \text{as} \quad \delta \to 0^+.$$

■

Proof of Lemma 5.2 parts 2), 3):

Let's begin with proving 2). Let u be as described there. Let $(\rho, y^\mu) = (x^i)$ be boundary coordinates of $\overline{M}$, consider the coordinates $v^i\partial/\partial x^i \mapsto (x^i, v^i)$ of TM, and let (b_i) be a frame for $\mathcal{E}$ over these coordinates. We write u component-wise as

$$u = u^{ij}\pi^*\frac{\partial}{\partial x^i} \otimes \pi^* b_j \qquad \text{and} \qquad u = {}^0u_i^j \pi_0^* \frac{dx^i}{\rho} \otimes \pi_0^* b_j. \tag{5.7}$$

We have that $u^{ij} = \rho^{-1}g^{ii'}\left({}^0u_{i'}^j\right)$ when evaluated at identified points. By (4.13) we have that

$$\mathbb{X}u \text{ over } SM = \left(X\left(u^{kj}\right) + \Gamma^k_{i'i}v^{i'}u^{ij}\right)\pi^*\frac{\partial}{\partial x^k} \otimes \pi^* b_j + u^{ij}\pi^*\frac{\partial}{\partial x^i} \otimes \left({}^{\mathcal{E}}\Gamma^k_{i'j}v^{i'}\pi^* b_k\right)$$

where Γ^k_{ij} and ${}^{\mathcal{E}}\Gamma^k_{ij}$ denote the Christoffel symbols and connection symbols of $\nabla^{\mathcal{E}}$ respectively. Hence, pulling the above to ${}^0N \otimes \pi_0^*\mathcal{E}$ gives

$$\mathbb{X}u \text{ over } {}^0S^*M\big|_M$$

$$= \left(X_0\left(\rho^{-1}g^{kk'}\left({}^0u_{k'}^j\right)\right) + \Gamma^k_{i'i}\rho^{-1}g^{i'i''}\overline{\eta}_{i''}\rho^{-1}g^{ii'''}\left({}^0u_{i'''}^j\right)\right)\left(\rho g_{kk'}\pi_0^*\frac{dx^{k'}}{\rho}\right) \otimes \pi_0^* b_j$$
$$+\rho^{-1}g^{ii'}\left({}^0u_{i'}^j\right)\left(\rho g_{ii''}\pi_0^*\frac{dx^{i''}}{\rho}\right) \otimes \left({}^{\mathcal{E}}\Gamma^k_{i'j}\rho^{-1}g^{i'i'''}\overline{\eta}_{i'''}\pi_0^* b_k\right).$$

Now, letting $\overline{\Gamma}^k_{ij}$ denote the smooth Christoffel symbols of the smooth metric $\overline{g} = \rho^2 g$, the conformal transformation law of Christoffel symbols (e.g. see Proposition 7.29 in [22]) gives that Γ^k_{ij} are ρ^{-1} times something smooth on $\overline{M}$. Hence recalling the form of X_0 from earlier, we get that the above is in indeed $\rho^\alpha C^\infty\left({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\right)$ as well. The equality in 2) follows the same way that we proved the equality in 1).

Finally, let's prove 3). Let u and w be as described there. First let's look at u whose components we write in our boundary coordinates and frame just as we did in (5.4). In the proof of Lemma 3.2 in [14] the authors give an equation for the vertical derivative[6] of u in terms of an operator "$\overset{\mathrm{v}}{\nabla}$" for which an explicit equation is given on page 350 of [37]. As the authors do in [14], we assume that (b_i) is orthonormal so that may use their formula to get that

$$\overset{\mathrm{v}}{\nabla}^{\pi^*\mathcal{E}} u \text{ over } SM = \partial^i u^j \pi^* \frac{\partial}{\partial x^i} \otimes \pi^* b_j, \tag{5.8}$$

where for any $f \in C^\infty(SM)$

$$\partial_i f := [\partial_{v^i}(f \circ p)]|_{SM},$$
$$\partial^i f := g^{ii'} \partial_i f,$$

where $p : TM \setminus \{0\} \to SM$ is the radial projection map $v \mapsto {}^{v}/_{|v|_g}$ over the tangent bundle minus the zero section. Consider the coordinates $\overline{\eta}_i {}^{dx^i}/_{\rho} \mapsto (x^i, \overline{\eta}_i)$ of ${}^0T^*\overline{M}$ and observe that the canonical identification $H^* \circ \flat : TM \to {}^0T^*\overline{M}\big|_M$ is given by $\overline{\eta}_i = \rho g_{ii'} v^{i'}$, whose differential takes

$$\frac{\partial}{\partial v^i} \longmapsto \rho g_{ii'} \frac{\partial}{\partial \eta_{i'}}.$$

Hence pulling the vertical derivative of u back to ${}^0N \otimes \pi_0^*\mathcal{E}$ gives

$$\overset{\mathrm{v}}{\nabla}^{\pi^*\mathcal{E}} u \text{ over } {}^0S^*M\big|_M = g^{ii'} \rho g_{i'i''} \left[\partial_{\overline{\eta}_{i''}} \left({}^0u^j \circ p_0\right)\right]\big|_{{}^0S^*\overline{M}} \left(\rho g_{ii'''} \pi_0^* \frac{dx^{i'''}}{\rho}\right) \otimes \pi_0^* b_j,$$

where $p_0 : {}^0T^*\overline{M} \setminus \{0\} \to {}^0S^*\overline{M}$ is the analogous map $\overline{\zeta} \mapsto \overline{\zeta}/|\overline{\zeta}|_{g_0}$. As before, we see that this is in $\rho^\alpha C^\infty\left({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\right)$.

Next let's take a look at w, whose components we write in our boundary coordinates and frames as in (5.7) but with "u" replaced with "w." Using (5.8) above, a straightforward generalization of the derivation of the equation for "$\overset{\mathrm{v}}{\mathrm{div}}\, Z$" given on page 352 of [37] gives

$$\overset{\mathrm{v}}{\mathrm{div}}^{\pi^*\mathcal{E}} w \text{ over } SM = \partial_i w^{ij} \pi^* b_j. \tag{5.9}$$

Hence, pulling this back to $\pi_0^*\mathcal{E}$ gives

$$\overset{\mathrm{v}}{\mathrm{div}}^{\pi^*\mathcal{E}} w \text{ over } {}^0S^*\overline{M}\big|_M = \rho g_{ii'} \left[\partial_{\overline{\eta}_{i'}} \left(\left[\rho^{-1} g^{ii''} \left({}^0w^j_{i''}\right)\right] \circ p_0\right)\right]\Big|_{{}^0S^*\overline{M}} \pi_0^* b_j.$$

This is indeed in $\rho^\beta C^\infty\left({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\right)$.

The equality in 3) follow essentially the same way we proved the analogous fact in 1). An example of a minor change that's needed is that the analog of (5.6) will be

$$\langle \overset{\mathrm{v}}{\nabla}^{\pi^*\mathcal{E}} u, \phi_\delta w \rangle_{L^2} = -\langle u, \phi_\delta \overset{\mathrm{v}}{\mathrm{div}}^{\pi^*\mathcal{E}} w \rangle_{L^2},$$

which we note doesn't have an analogous "third term" as in (5.6) because ϕ_δ only depends on position and thus isn't affected by the vertical divergence. From there you proceed as before.

[6] We remark that in their work they write "$\overset{\mathrm{v}}{\nabla}^{\mathcal{E}}$" for what we denote by "$\overset{\mathrm{v}}{\nabla}^{\pi^*\mathcal{E}}$."

$\blacksquare$

For use in Section 5.2 below, we record the decay preserving property of the horizontal derivative:

Lemma 5.10. *If* $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$*, then* $\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ *as well.*

Proof: Take any such u. Let $(\rho, y^\mu) = (x^i)$ be boundary coordinates of $\overline{M}$, consider the coordinates $v^i\partial/\partial x^i \mapsto (x^i, v^i)$ of TM, and let (b_i) be an orthonormal frame for $\mathcal{E}$ over these coordinates. Let ${}^{\mathcal{E}}\Gamma_{ij}^k$ denote the connection symbols of $\nabla^{\mathcal{E}}$ with respect to these coordinates and frame. We write u component-wise as in (5.4). By the equations for the horizontal and vertical derivatives given in the proof of Lemma 3.2 in [14] and on page 350 of [37],

$$\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} u \text{ over } SM = \big(\delta^i u^j - \big(v^k \delta_k u^j\big) v^i\big) \pi^* \frac{\partial}{\partial x^i} \otimes \pi^* b_j + u^l \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} \big({}^{\mathcal{E}}\Gamma_{kl}^j v^k \pi^* b_j\big). \tag{5.11}$$

where for any $f \in C^\infty(SM)$

$$\delta_i f := \big[\big(\partial_{x^i} - \Gamma_{ij}^k v^j \partial_{v^k}\big)(f \circ p)\big]\big|_{SM},$$

$$\delta^i f := g^{ii'} \delta_{i'} f,$$

where $p : TM \setminus \{0\} \to SM$ is the radial projection map $v \mapsto v/|v|_g$. Considering that we already know that $\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}$ preserves decay rates at infinity, the lemma follows by pulling (5.11) back to ${}^0N \otimes \pi_0^*\mathcal{E}$ as before - details are left to the reader.

$\blacksquare$

Next we need the following lemma that tells us that the curvature operators also don't decrease decay rates at infinity.

Lemma 5.12. *Suppose that* $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$ *and* $w \in \rho^\beta C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ *for some real numbers* $\alpha, \beta \in \mathbb{R}$*. Then* $Ru \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$ *and* $F^{\mathcal{E}} w \in \rho^{\beta+2} C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$.

Proof: Let $(\rho, y^1, \ldots, y^n) = (x^i)$ be boundary coordinates of $\overline{M}$ and let (b_i) be a frame for $\mathcal{E}$ over these coordinates. We write u component-wise as in (5.7) and w as in (5.4) but with "u" replaced with "w." We have that

$$Ru \text{ over } SM = \big(\partial_{x^i}\Gamma_{j'k}^l - \partial_{x^{j'}}\Gamma_{ik}^l + \Gamma_{j'k}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{j'm}^l\big) u^{ij} v^{j'} v^k \pi^* \frac{\partial}{\partial x^l} \otimes \pi^* b_j.$$

Letting $R_{ij'k}{}^l$ denote the quantity in $(\ldots)$ and pulling this back to ${}^0N \otimes \pi_0^*\mathcal{E}$ gives

$$Ru \text{ over } {}^0S^*\overline{M}\big|_M = R_{ij'k}{}^l \rho^{-1} g^{ii'} \big({}^0u_{i'}^j\big) \rho^{-1} g^{j'j''} \overline{\eta}_{j''} \rho^{-1} g^{kk'} \overline{\eta}_{k'} \left(\rho g_{ll'} \pi_0^* \frac{dx^{l'}}{\rho}\right) \otimes \pi_0^* b_j.$$

Since Γ_{ij}^k are ρ^{-1} times something smooth on $\overline{M}$, it's easy to see that each $R_{ij'k}{}^l$ is ρ^{-2} times something smooth on $\overline{M}$. From this we see that this is indeed in $\rho^\alpha C^\infty\big({}^0S^*\overline{M}; {}^0N \otimes \pi_0^*\mathcal{E}\big)$.

Next, looking at (4.16) and (4.17) we have that

$$F^{\mathcal{E}}w \text{ over } SM = P_{N\otimes\pi^*\mathcal{E}}\left(w^l g^{jj'} f_{ij}{}^k{}_l v^i \pi^* \frac{\partial}{\partial x^{j'}} \otimes \pi^* b_k\right),$$

and so

$$F^{\mathcal{E}}w \text{ over } {}^0S^*\overline{M}\big|_M = P_{{}^0N\otimes\pi_0^*\mathcal{E}}\left({}^0w^l g^{jj'} f_{ij}{}^k{}_l \rho^{-1} g^{ii'} \overline{\eta}_{i'}\left(\rho g_{j'j''}\pi_0^* \frac{dx^{j''}}{\rho}\right) \otimes \pi_0^* b_k\right),$$

where $P_{{}^0N\otimes\pi_0^*\mathcal{E}}$ is an analogous map that projects the first component of a tensor product onto ${}^0N$. From here we see that is in $\rho^\beta C^\infty\left({}^0S^*\overline{M}; {}^0N\otimes\pi_0^*\mathcal{E}\right)$.

■

We need one final lemma that provides the needed commutator formulas to prove Theorem 5.1. The following lemma is Lemma 3.2 in [14], where one can also find a proof.

Lemma 5.13. *The following are true, where* $[\ldots,\ldots]$ *denotes the commutator bracket.*

$$\left[\mathbb{X}, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\right] = -\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}, \tag{5.14}$$

$$\left[\mathbb{X}, \overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}\right] = R\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} + F^{\mathcal{E}}, \tag{5.15}$$

$$\overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} - \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} = n\mathbb{X}, \tag{5.16}$$

$$\left[\mathbb{X}, \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\right] = -\overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}. \tag{5.17}$$

Proof of Theorem 5.1:

Let u be as described in the theorem. By Lemma 5.2 we have that

$$\langle \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u\rangle_{L^2} = \langle \mathbb{X}\overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u, u\rangle_{L^2}.$$

We get that this is equal to (see right after for justifications)

$$\begin{aligned}
&\langle -\overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u\rangle_{L^2},\\
&= \langle -\overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\mathbb{X}\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\mathbb{X}\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, u\rangle_{L^2},\\
&= \langle -\overset{\mathrm{h}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\left(R\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} + F^{\mathcal{E}}\right)u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\mathbb{X}\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, u\rangle_{L^2},\\
&= \langle -n\mathbb{X}\mathbb{X}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\left(R\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} + F^{\mathcal{E}}\right)u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\mathbb{X}\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, u\rangle_{L^2},
\end{aligned}$$

where in the above four lines we used respectively (5.17), (5.14), (5.15), and (5.16). Applying Lemma 5.2 again gives that this is equal to

$$n\langle \mathbb{X}u, \mathbb{X}u\rangle_{L^2} + \langle \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}R\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u + \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}F^{\mathcal{E}}u, u\rangle_{L^2} + \langle \mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, \mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2}.$$

We can split the second inner product over the "+" sign since by Lemma 5.12, R and $F^{\mathcal{E}}$ don't affect decay rates at the boundary "$\partial\overline{M}$." Doing this and then applying Lemma 5.2 to the resultant middle two terms proves the theorem.

■

5.2 Finite Degree of Solutions to Transport Equations

In the proof of Theorem 2.6 we will end up showing that the gauge satisfies an equation of the form

$$\mathbb{X}u + \Phi u = f,$$

which is also called a "transport equation." It turns out that this equation has good behavior with respect to vertical Fourier analysis, which we now introduce. Consider the **vertical Laplacian**:

$$\Delta^{\pi^*\mathcal{E}} = -\overset{\mathrm{v}}{\operatorname{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}} : C^\infty(SM;\pi^*\mathcal{E}) \longrightarrow C^\infty(SM;\pi^*\mathcal{E}).$$

Let's see what this looks like in coordinates. Let (x^i) be coordinates of M, let (r_i) be an orthonormal frame of TM over their domain, let (b_j) denote a frame of $\mathcal{E}$ over their domain, and consider the coordinates

$$v^i r_i \longmapsto \left(x^i, v^i\right) \tag{5.18}$$

of TM. Then we claim that for any smooth section $u = u^j\pi^*b_j$,

$$\Delta^{\pi^*\mathcal{E}}u = \left(-\Delta^{\mathbb{S}^n}u^j\right)\pi^*b_j$$

where "$-\Delta^{\mathbb{S}^n}$" is the negative Laplacian on the n-sphere in the variables v^i. This is most easily seen as follows. Pick an arbitrary point $x_0 \in M$ in the domain of our coordinates, choose normal coordinates $(\hat{x}^i)$ of M centered at x_0, and consider the coordinates $\hat{v}^i\partial/\partial\hat{x}^i \mapsto (\hat{x}^i, \hat{v}^i)$ of TM. Then observe that (5.8) and (5.9) tell us that on the sphere $S_{x_0}M$, the operator $\Delta^{\pi^*\mathcal{E}}$ applied to $u = \hat{u}^j\pi^*\partial/\partial x^j$ is given by $\left(-\Delta^{\mathbb{S}^n}\hat{u}^j\right)\pi^*\partial/\partial x^j$. The claim then follows by pushing this expression through the change of variables $(\hat{x}^i, \hat{v}^i) \mapsto (x^i, v^i)$.

From this observation and the theory of spherical harmonics (c.f. Section 2.H in [11] for the latter), we get several important implications. First, we get that the eigenvalues of $\Delta^{\pi^*\mathcal{E}}$ match that of $-\Delta^{\mathbb{S}^n}$:

$$\lambda_m = m(m+n-1) \quad \text{for integers} \quad m \geq 0.$$

Furthermore, letting Ω_m denote the set of smooth eigenfunctions of $\Delta^{\pi^*\mathcal{E}}$ with eigenvalue λ_m, any $u \in C^\infty(SM;\pi^*\mathcal{E})$ can be decomposed as (the pointwise converging) "Fourier series"

$$u = \sum_{m=0}^{\infty} u_m, \qquad u \in \Omega_m.$$

The u_m's are called u's Fourier modes. The maximum index m for which $u_m \neq 0$ is called the **degree** of u and is denoted by "$\deg u$" (which could be infinity). Naturally, we say that u is of **finite degree** if its degree is finite. We can write an explicit equation for the Fourier modes as follows. For each $m \in \mathbb{Z}_+$ we let

$$\{Y_k^m : k = 1, \ldots, l_m\}$$

denote a real-valued basis of eigenfunctions of $-\Delta^{\mathbb{S}^n}$ with eigenvalue λ_m. Then in the coordinates (5.18) and frame (b_i) there,

$$u_m^j\left(x^i, v^i\right) = \sum_{k=1}^{l_m}\left[\int_{\mathbb{S}^n} u^j\left(x^i, w^i\right)Y_k^m\left(w^i\right)dw_{\mathbb{S}^n}\right]Y_k^m\left(v^i\right). \tag{5.19}$$

Lemma 5.20. *Suppose that $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M};\pi_0^*\mathcal{E}\big)$. Then each $u_m \in \rho^\alpha C^\infty\big({}^0S^*\overline{M};\pi_0^*\mathcal{E}\big)$ as well.*

Proof: Consider the coordinates (5.18) and frame (b_i) there. Suppose furthermore that the frame (r_i) there was obtained by mapping a local orthonormal frame $\left(\overline{\zeta}^i\right)$ of ${}^0T^*\overline{M}$ via the canonical identification $\sharp \circ H : {}^0T^*\overline{M}\big|_M \to TM$. Considering the coordinates $\overline{\eta}_i\overline{\zeta}^i \mapsto (x^i, \overline{\eta}_i)$ of ${}^0T^*\overline{M}$, we get that

$$u_m^j \text{ over } {}^0S^*\overline{M}\big|_M = u_m^j\big(x^i, \overline{\eta}_i\big) = \sum_{k=1}^{l_m}\left[\int_{\mathbb{S}^n} u^j\big(x^i, w^i\big)Y_k^m\big(w^i\big)dw_{\mathbb{S}^n}\right]Y_k^m(\overline{\eta}_i).$$

From here the lemma follows right away if one chooses (x^i) to be the interior of boundary coordinates of $\overline{M}$.

■

Lemma 5.21. *The spaces $\Omega_m \cap L^2(SM;\pi^*\mathcal{E})$ and $\Omega_{m'} \cap L^2(SM;\pi^*\mathcal{E})$ are orthogonal with respect to L^2 when $m \neq m'$.*

The above lemma follows from the theory of spherical harmonics and the fact that integrals over SM can be partitioned as described in Section 4.4 above.

One of the nice properties of $\mathbb{X}$ is that it maps

$$\mathbb{X} : \Omega_m \longrightarrow \Omega_{m-1} \oplus \Omega_{m+1}. \tag{5.22}$$

This is proven in Section 3.4 of [14]. Similarly, multiplication on the left by Φ maps $\Omega_m \to \Omega_m$ since Φ has no dependence on the vertical variable "v." In particular, we see that the operator in the transport equation "$\mathbb{X} + \Phi$" maps sections of finite degree to sections of finite degree. The converse is also true (recall our standing assumption that Φ is skew-Hermitian):

Theorem 5.23. *Assume that the sectional curvatures of g are negative. If $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M};\pi^*\mathcal{E}\big)$, where $\alpha \geq {}^{(n+1)}\!/_2$, solves*

$$\mathbb{X}u + \Phi u = f \tag{5.24}$$

for some $f \in C^\infty\big({}^0S^*\overline{M};\pi^*\mathcal{E}\big)$ of finite degree, then u is also of finite degree.

To prove this, we need several preliminary results.

Lemma 5.25. *It holds that*

$$\big[\mathbb{X}, \Delta^{\pi^*\mathcal{E}}\big] = 2\overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}} + n\mathbb{X}.$$

The above lemma is stated as Lemma 3.4 of [14], whose proof is essentially identical to that of Lemma 3.5 in [37]. To state the next preliminary result we observe that because of (5.22), over each Ω_m we can decompose $\mathbb{X} = \mathbb{X}_- + \mathbb{X}_+$ where

$$\mathbb{X}_\pm : \Omega_m \longrightarrow \Omega_{m\pm1}.$$

We point out that the maps $\mathbb{X}_\pm$ are distinct for different Ω_m even though we use the same notation to denote them. Furthermore, $\mathbb{X}_\pm$ have the same decay rate preserving properties as $\mathbb{X}$ described in Lemma 5.2 because of Lemma 5.20 above. We mention that the idea of splitting the action of the geodesic vector field as above was first introduced by Guillemin and Kazhdan - see [15]. The following preliminary result is a special case of the Pestov identity with a connection (Theorem 5.1).

Proposition 5.26. *Suppose that $u \in \Omega_m \cap \rho^\alpha C^\infty\left({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\right)$ for some $\alpha \geq {}^{(n+1)}/_2$. Then*

$$(2m+n)\|\mathbb{X}_+u\|_{L^2}^2$$
$$= \|\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2 + (2m+n-2)\|\mathbb{X}_-u\|_{L^2}^2 - \langle R\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} - \langle F^{\mathcal{E}}u, \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2}.$$

Proof: We have that u satisfies the equation in Theorem 5.1. Let's take a look at the term

$$\begin{aligned}\|\mathbb{X}\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2 &= \langle -\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}u + \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u, -\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}u + \overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u\rangle_{L^2} && \text{by (5.14)},\\ &= \|\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2 + 2\langle \mathbb{X}u, \overset{\mathrm{v}}{\mathrm{div}}{}^{\pi^*\mathcal{E}}\overset{\mathrm{h}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} + \|\overset{\mathrm{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}u\|_{L^2}^2 && \text{Lemma 5.2 3) and Lemma 5.10.}\end{aligned}$$

Applying Lemma 5.25, we see that the middle term in the last quantity is equal to

$$\langle \mathbb{X}u, \mathbb{X}\Delta^{\pi^*\mathcal{E}}u - \Delta^{\pi^*\mathcal{E}}\mathbb{X}u - n\mathbb{X}u\rangle_{L^2}.$$

Splitting $\mathbb{X}u = \mathbb{X}_-u + \mathbb{X}_+u \in \Omega_{m-1} \oplus \Omega_{m+1}$, using that

$$\Delta^{\pi^*\mathcal{E}}u = \lambda_m u, \qquad \Delta^{\pi^*\mathcal{E}}\mathbb{X}_-u = \lambda_{m-1}\mathbb{X}_-u, \qquad \Delta^{\pi^*\mathcal{E}}\mathbb{X}_+u = \lambda_{m+1}u,$$

using Lemma 5.21, and then plugging the result into the equation in Theorem 5.1 proves the proposition after several cancellations.

■

The following lemma provides the contraction property that's needed in the proof of Theorem 5.23.

Lemma 5.27. *Suppose that the sectional curvatures of g are negative. Then for any $\alpha \geq {}^{(n+1)}/_2$ there exist real constants $c_m \to \infty$ such that for sufficiently large m,*

$$\begin{cases} \|\mathbb{X}_-u\|_{L^2}^2 + c_m\|u\|_{L^2}^2 \leq \|\mathbb{X}_+u\|_{L^2}^2 & \text{if} \quad n \neq 2, \\ \|\mathbb{X}_-u\|_{L^2}^2 + c_m\|u\|_{L^2}^2 \leq d_m\|\mathbb{X}_+u\|_{L^2}^2 & \text{if} \quad n = 2, \end{cases} \tag{5.28}$$

for all $u \in \Omega_m \cap \rho^\alpha C^\infty(SM; \pi^*\mathcal{E})$ where $d_m = 1 + {}^1/_{[(2m-1)(m+1)^2]}$.

Proof: We begin by using the fact that the sectional curvatures of g tend to -1 at $\partial\overline{M}$. Precisely, by the remark after Proposition 1.10 in [27] there exists an $\varepsilon > 0$ so the sectional curvatures of g are less than $-\kappa'$ for some $\kappa' > 0$ over the region $\{\rho < \varepsilon\}$. Hence this, the compactness of $\{\rho \geq \varepsilon\}$, and the negative curvature assumption imply that there exists a $\kappa > 0$ such that the sectional curvatures of g are bounded above by $-\kappa$ on all of M.

Now, take any α and u as in the statement of the lemma. We have that u satisfies the equation in Proposition 5.26 above. We begin by estimating the L^2 norm of the term $\overset{\text{h}}{\nabla}{}^{\pi^*\mathcal{E}}u$ by utilizing the trick of looking at its vertical divergence. By Lemma 5.25 we have that

$$\overset{\text{v}}{\text{div}}{}^{\pi^*\mathcal{E}}\overset{\text{h}}{\nabla}{}^{\pi^*\mathcal{E}}u = \frac{1}{2}\mathbb{X}\Delta^{\pi^*\mathcal{E}}u - \frac{1}{2}\Delta^{\pi^*\mathcal{E}}\mathbb{X}u - \frac{n}{2}\mathbb{X}u,$$
$$= \frac{1}{2}(\mathbb{X}_+\lambda_m u + \mathbb{X}_-\lambda_m u) - \frac{1}{2}(\lambda_{m+1}\mathbb{X}_+u + \lambda_{m-1}\mathbb{X}_-u) - \frac{n}{2}(\mathbb{X}_+u + \mathbb{X}_-u),$$
$$= -(m+n)\mathbb{X}_+u + (m-1)\mathbb{X}_-u,$$
$$= -\overset{\text{v}}{\text{div}}{}^{\pi^*\mathcal{E}}\left(-\frac{m+n}{\lambda_{m+1}}\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}_+u + \frac{m-1}{\lambda_{m-1}}\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}_-u\right).$$

Plugging the expression for the λ_k's into this, we conclude that

$$\overset{\text{h}}{\nabla}{}^{\pi^*\mathcal{E}}u = \frac{1}{m+1}\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}_+u - \frac{1}{m+n-2}\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}\mathbb{X}_-u + Z$$

where $Z \in C^\infty(SM; N \otimes \pi^*\mathcal{E})$ is such that $\overset{\text{v}}{\text{div}}{}^{\pi^*\mathcal{E}}Z = 0$ and hence perpendicular to the other two terms on the right-hand side with respect to L^2. Using Lemma 5.21 and Lemma 5.2 3), this quickly gives us the L^2 estimate

$$\|\overset{\text{h}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2 \geq \frac{m+n}{m+1}\|\mathbb{X}_+u\|_{L^2}^2 + \frac{m-1}{m+n-2}\|\mathbb{X}_-u\|_{L^2}^2.$$

Now, plugging this into the equation in Proposition 5.26 gives

$$\left(2m+n-\frac{m+n}{m+1}\right)\|\mathbb{X}_+u\|_{L^2}^2 \tag{5.29}$$
$$\geq \left(2m+n-2+\frac{m-1}{m+n-2}\right)\|\mathbb{X}_-u\|_{L^2}^2 - \langle R\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, \overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} - \langle F^{\mathcal{E}}u, \overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2}.$$

We have that the term

$$-\langle R\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u, \overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} \geq \kappa\|\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\|_{L^2}^2 = \kappa\lambda_m\|u\|_{L^2}^2.$$

Furthermore,

$$-\langle F^{\mathcal{E}}u, \overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\rangle_{L^2} \geq -\|F^{\mathcal{E}}\|_{L^\infty}\|u\|_{L^2}\|\overset{\text{v}}{\nabla}{}^{\pi^*\mathcal{E}}u\| = -\|F^{\mathcal{E}}\|_{L^\infty}\lambda_m^{1/2}\|u\|_{L^2}^2.$$

Hence, letting $a_{m,n}$ and $b_{m,n}$ denote the coefficients of $\|\mathbb{X}_+u\|_{L^2}^2$ and $\|\mathbb{X}_-u\|_{L^2}^2$ in (5.29) above respectively we get that

$$\frac{a_{m,n}}{b_{m,n}}\|\mathbb{X}_+u\|_{L^2}^2 \geq \|\mathbb{X}_-u\|_{L^2}^2 + \frac{\kappa\lambda_m - \|F^{\mathcal{E}}\|_{L^\infty}\lambda_m^{1/2}}{b_{m,n}}\|u\|_{L^2}^2.$$

Elementary algebra shows that $a_{m,n}/b_{m,n}$ is less than or equal to 1 if $n \neq 2$ and $m > 1$, and is equal to d_m if $n = 2$. Since $\lambda_m = O(m^2)$, the lemma follows.

$\blacksquare$

We need one last technical lemma:

Lemma 5.30. *If $u \in \rho^\alpha C^\infty\big({}^0S^*\overline{M}; \pi_0^*\mathcal{E}\big)$ for $\alpha \geq {}^{(n+1)}/_2$, then $\|\mathbb{X}_+ u_m\|_{L^2} \to 0$ as $m \to \infty$.*

Proof: Let $(W, (x^i))$ be coordinates of $\overline{M}$ (W denotes the domain), let (b_j) denote an *orthonormal* frame of $\mathcal{E}$ over W, let (r_i) be an orthonormal frame of TM over $W \cap M$, and consider the coordinates $v^i r_i \mapsto (x^i, v^i)$ of TM. We will show that $\|\mathbb{X}_+ u_m\|_{L^2(\pi^{-1}[W])} \to 0$, from which the lemma will follow by covering $\overline{M}$ by a finite number of such sets W. Letting ${}^{\mathcal{E}}\Gamma_{ij}^k$ denote the connection symbols of $\nabla^{\mathcal{E}}$ with respect to (r_i) and (b_j), we have that

$$\|\mathbb{X}_+ u_m\|_{L^2(\pi^{-1}[W])} \leq \|\mathbb{X} u_m\|_{L^2(\pi^{-1}[W])}$$

$$= \sum_{k=1}^{\operatorname{rank}\mathcal{E}} \int_{W^0} \int_{\mathbb{S}^n} \left| X u_m^k(x^i, v^i) + {}^{\mathcal{E}}\Gamma_{ij}^k(x^i) v^i u_m^j(x^i, v^i) \right|^2 dv_{\mathbb{S}^n} dx$$

where $W^0 = W \cap M$. Recall that by the Cauchy-Schwarz inequality

$$|a_1 + \ldots + a_k|^2 \leq k\big(|a_1|^2 + \ldots + |a_k|^2\big), \tag{5.31}$$

where the a_i's are complex numbers. Then, using that $\left|{}^{\mathcal{E}}\Gamma_{ij}^k v^i\right| < C$ on W for some $C > 0$ independent of u, we get that the previous quantity is bounded by

$$C' \sum_{k=1}^{\operatorname{rank}\mathcal{E}} \left[\int_{W^0} \int_{\mathbb{S}^n} \left| X u_m^k \right|^2 dv_{\mathbb{S}^n} dx + \int_{W^0} \int_{\mathbb{S}^n} \left| u_m^k \right|^2 dv_{\mathbb{S}^n} dx \right]$$

for some $C' > 0$ also independent of u. We complete the proof by showing that as $m \to \infty$,

1. $\int_{W^0} \int_{\mathbb{S}^n} \left| X u_m^k \right|^2 dv_{\mathbb{S}^n} dx \to 0$,
2. $\int_{W^0} \int_{\mathbb{S}^n} \left| u_m^k \right|^2 dv_{\mathbb{S}^n} dx \to 0$.

Let's start by proving 2). Taking (5.19), writing $Y_k^m = ({}^1/_{\lambda_m})^\mu \big(-\Delta^{\mathbb{S}^n}\big)^\mu Y_k^m$ inside the integral there for some integer μ to be determined later, then integrating by parts gives

$$\left| u_m^k(x^i, v^i) \right| = \left| \left(\frac{1}{\lambda_m} \right)^\mu \sum_{k=1}^{l_m} \left[\int_{\mathbb{S}^n} \big(-\Delta^{\mathbb{S}^n} u_m^k\big)(x^i, w^i) Y_k^m(w^i) dw_{\mathbb{S}^n} \right] Y_k^m(v^i) \right|$$

$$\leq \frac{\sup_{S_x M} \left| \left[\Delta^{\pi^*\mathcal{E}}\right]^\mu u_m \right|}{\lambda_m^\mu} \sum_{k=1}^{l_m} \sup |Y_k^m|^2.$$

By Lemma 5.2, $\left[\Delta^{\pi^*\mathcal{E}}\right]^\mu u_m \in \rho^\alpha C^\infty({}^0S^*M; \pi^*\mathcal{E})$. From the theory of spherical harmonics, we have that the sum "$\sum \ldots$" on the right-hand side is bounded by a polynomial in m (e.g. see Corollary 2.56 and Theorem 2.57 (a) and (f) in [11] and consider $Z_k^x(x)$ there). Hence 2) follows by choosing μ big enough.

Finally, let's prove 1). We have that

$$Xu_m^k = v^{i'} r_{i'} u_m^k - v^{i'} v^{j'} \Gamma_{i'j'}^{k'} \frac{\partial}{\partial v^{k'}} u_m^k \tag{5.32}$$

where Γ_{ij}^k are the Christoffel symbols of g with respect to (r_i). Since $SM = (v^1)^2 + \ldots + (v^n)^2$ in our coordinates, $\Gamma_{i'j'}^{k'} \partial/\partial v^{k'}$ is tangent to $S_x M$ over all $x \in U^0$. Hence by the Cauchy-Schwarz inequality the term

$$\left| \Gamma_{i'j'}^{k'} \frac{\partial}{\partial v^{k'}} u_m^k \right| \leq \sqrt{\left(\Gamma_{ij}^1\right)^2 + \ldots + \left(\Gamma_{ij}^{n+1}\right)^2} \left| \mathrm{grad}_{\mathbb{S}^n} u_m^k \right|$$

where $\mathrm{grad}_{\mathbb{S}^n}$ is Euclidean spherical gradient in terms of $\partial/\partial v^j$ and $|\ldots|$ is the Euclidean length. Thus plugging (5.19) into the first u_m^k on the right-hand side of (5.32) and using the above estimate gives that for some $C'' > 0$ independent of u,

$$\frac{1}{C''} \left| Xu_m^k \right|^2 \leq \left| \sum_{k=1}^{l_m} \left[v^{i'} \int_{\mathbb{S}^n} r_{i'} u_m^k (x^i, w^i) Y_k^m (w^i) dw_{\mathbb{S}^n} \right] Y_k^m (v^i) \right|$$
$$+ \left| \sum_{k=1}^{l_m} \left[\int_{\mathbb{S}^n} u_m^k (x^i, w^i) Y_k^m (w^i) dw_{\mathbb{S}^n} \right] \mathrm{grad}_{\mathbb{S}^n} Y_k^m (v^i) \right|$$

If $(x^i) = (\rho, y^\mu)$ were boundary coordinates, suppose we required them to be asymptotic boundary normal coordinates as in Section 2.2 and that we constructed the orthonormal frame (r_i) to be of the specific form

$$r_0 = \rho \frac{\partial}{\partial \rho}, \qquad r_\mu = \rho \widetilde{r}_\mu,$$

where (r_μ) is an orthonormal frame spanned by $(\partial/\partial y^\mu)$. This way, we see that the terms $r_{i'} u_m^k \in \rho^\alpha C^\infty \left({}^0S^* \overline{M}; \pi_0^* \mathcal{E} \right)$ in the above inequality. Then using the inequality (5.31) again and the fact that $\|\mathrm{grad}_{\mathbb{S}^n} Y_k^m\|_{L^2(\mathbb{S}^n)}^2 = \langle -\Delta^{\mathbb{S}^n} Y_k^m, Y_k^m \rangle_{L^2(\mathbb{S}^n)} = \lambda_m$, doing a similar thing as above we get that for some $C''' > 0$ independent of u,

$$\int_{\mathbb{S}^n} \left| Xu_m^k \right|^2 dv_{\mathbb{S}^n} \leq C''' \left(\sum_{i'=1}^{n+1} \left(\frac{\sup_{S_x M} \left| \left[\Delta^{\pi^* \mathcal{E}} \right]^\mu r_{i'} u_m \right|}{\lambda_m^\mu} \right) + \frac{\sup_{S_x M} \left| \left[\Delta^{\pi^* \mathcal{E}} \right]^\mu u_m \right|}{\lambda_m^{\mu - 1}} \right) \sum_{k=1}^{l_m} \sup \left| Y_k^m \right|^2$$

where $r_{i'} u_m = r_{i'} u^j \pi^* b_j$. Hence the lemma follows again by choosing μ big enough.

■

Proof of Theorem 5.23:

We start by assuming that $n \neq 2$ since the proof of the case $n = 2$ requires a slight modification. Comparing Fourier modes of order $m > \deg f$ in (5.24) gives that

$$\mathbb{X}_+ u_{m-1} + \mathbb{X}_- u_{m+1} + \Phi u_m = 0, \tag{5.33}$$

where we note that comparing Fourier modes is justified by the theory of spherical harmonics. Using this relation and plugging u_{m+1} into u in (5.28) gives that

$$\|\mathbb{X}_+ u_{m+1}\|_{L^2}^2 \geq \|\mathbb{X}_+ u_{m-1}\|_{L^2}^2 + \|\Phi u_m\|_{L^2}^2 + c_{m+1}\|u_{m+1}\|_{L^2}^2 + 2\mathrm{Re}\langle \Phi u_m, \mathbb{X}_+ u_{m-1}\rangle_{L^2}. \quad (5.34)$$

The idea of the proof is the following. One can plug the relation (5.33) with "$m+1$" replaced by "$m-1$" into $\|\mathbb{X}_+ u_{m-1}\|_{L^2}^2$ here, use (5.28) again, and then proceed recursively in the same way. One will get a long expression on the right which one needs to cleverly manipulate to bound $\|u_{m_0}\|_{L^2}^2$ for some fixed index m_0. Then using that $\|\mathbb{X}_+ u_{m+1}\|_{L^2}^2$ goes to zero as $m \to \infty$ will force $\|u_{m_0}\|_{L^2}^2 = 0$. The first obstacle to accomplishing this is the inner product term $2\mathrm{Re}\langle \Phi u_m, \mathbb{X}_+ u_{m-1}\rangle_{L^2}$. The following claim helps resolve this.

To state the claim, we introduce the following notation. For any $U \in C^\infty(SM; \operatorname{End} \pi^*\mathcal{E})$ (such as Φ up to identification), we let $\mathbb{X}U \in C^\infty(SM; \operatorname{End} \pi^*\mathcal{E})$ denote the unique endomorphism field satisfying $(\mathbb{X}U)h = [\mathbb{X}, U]h$ for any $h \in C^\infty(SM; \pi^*\mathcal{E})$. We leave it to the reader to show that $\mathbb{X}U$ is well-defined (or see Section 5.4 below) and that this operator $\mathbb{X}$ preserves decay rates at infinity as in Lemma 5.2 using the ideas from its proof.

Claim: The following identity is true (because Φ is skew-Hermitian):

$$\langle \mathbb{X}_+ u_{m-1}, \Phi u_m\rangle_{L^2} + \overline{\langle \mathbb{X}_+ u_{m-2}, \Phi u_{m-1}\rangle}_{L^2} = -\langle u_{m-1}, (\mathbb{X}\Phi) u_m\rangle_{L^2} - \|\Phi u_{m-1}\|_{L^2}^2.$$

Proof of claim: This is simply a computation:

$$\begin{aligned}
\langle \mathbb{X}_+ u_{m-1}, \Phi u_m\rangle_{L^2} &= \langle \mathbb{X} u_{m-1}, \Phi u_m\rangle_{L^2} && \text{use } \mathbb{X} = \mathbb{X}_- + \mathbb{X}_+ \text{ and Lemma 5.21,}\\
&= -\langle u_{m-1}, \mathbb{X}(\Phi u_m)\rangle_{L^2} && \text{Lemma 5.2,}\\
&= -\langle u_{m-1}, (\mathbb{X}\Phi) u_m + \Phi \mathbb{X} u_m\rangle_{L^2} && \text{definition of } \mathbb{X}\Phi,\\
&= -\langle u_{m-1}, (\mathbb{X}\Phi) u_m\rangle_{L^2} - \langle u_{m-1}, \Phi \mathbb{X}_- u_m\rangle_{L^2} && \mathbb{X} = \mathbb{X}_- + \mathbb{X}_+ \text{ and Lemma 5.21,}\\
-\langle u_{m-1}, (\mathbb{X}\Phi) u_m\rangle_{L^2} &+ \langle u_{m-1}, \Phi[\Phi u_{m-1} + \mathbb{X}_+ u_{m-2}]\rangle_{L^2} && \text{used (5.33).}
\end{aligned}$$

From here the claim follows by rearranging and using that Φ is skew-Hermitian.

End of proof of claim.

We return to proving the theorem. The above claim tells us that to get rid of the last term in (5.34), we can add (5.34) and (5.34) with "m" replaced by "$m-1$":

$$\|\mathbb{X}_+ u_{m+1}\|_{L^2}^2 + \|\mathbb{X}_+ u_m\|_{L^2}^2 \geq \|\mathbb{X}_+ u_{m-1}\|_{L^2}^2 + \|\mathbb{X}_+ u_{m-2}\|_{L^2}^2 + \|\Phi u_m\|_{L^2}^2 + \|\Phi u_{m-1}\|_{L^2}^2$$

$$+ c_{m+1}\|u_{m+1}\|_{L^2}^2 + c_m\|u_m\|_{L^2}^2 + 2\mathrm{Re}\langle \Phi u_m, \mathbb{X}_+ u_{m-1}\rangle_{L^2} + 2\mathrm{Re}\langle \Phi u_{m-1}, \mathbb{X}_+ u_{m-2}\rangle_{L^2},$$

and then apply the equation in the claim to get the following inequality, where for brevity $a_m = \|\mathbb{X}_+ u_m\|_{L^2}^2 + \|\mathbb{X}_+ u_{m-1}\|_{L^2}^2$:

$$a_{m+1} \geq a_{m-1} + \|\Phi u_m\|_{L^2}^2 + \|\Phi u_{m-1}\|_{L^2}^2 + c_{m+1}\|u_{m+1}\|_{L^2}^2 + c_m\|u_m\|_{L^2}^2$$

$$-2\mathrm{Re}\langle u_{m-1}, (\mathbb{X}\Phi) u_m\rangle_{L^2} - 2\|\Phi u_{m-1}\|_{L^2}^2$$

$$\geq a_{m-1} + c_{m+1}\|u_{m+1}\|_{L^2}^2 + c_m\|u_m\|_{L^2}^2 - \|u_{m-1}\|_{L^2}^2 - \|(\mathbb{X}\Phi) u_m\|_{L^2}^2 - \|\Phi u_{m-1}\|_{L^2}^2.$$

By the continuity of $\mathbb{X}\Phi$ and Φ over the compact manifold ${}^0S^*\overline{M}$, there exist constants $B, C > 0$ such that

$$\|\Phi h\|_{L^2}^2 \leq B\|h\|_{L^2}^2,$$
$$\|(\mathbb{X}\Phi)h\|_{L^2}^2 \leq C\|h\|_{L^2}^2,$$

for any $h \in C^\infty(SM; \pi^*\mathcal{E}) \cap L^2(SM; \pi^*\mathcal{E})$. Hence

$$a_{m+1} \geq a_{m-1} + r_m \tag{5.35}$$

where

$$r_m := c_{m+1}\|u_{m+1}\|_{L^2}^2 + (c_m - C)\|u_m\|_{L^2}^2 - (1+B)\|u_{m-1}\|_{L^2}^2.$$

Applying (5.35) recursively gives

$$a_{m+1} = a_{m_0-1} + r_{m_0} + r_{m_0+2} + \ldots + r_m$$

for any pair of indices $m, m_0 > \deg f$ such that $m = m_0 + 2k$ for some integer $k \geq 0$. We seek to bound the resultant tail of r_i's. To do so, choose m_0 big enough so that for $m \geq m_0$, c_m is bigger than both C and $B+1$. Hence if $m = m_0 + 2k$,

$$\begin{aligned} r_{m_0} + r_{m_0+2} + \ldots + r_m &= c_{m+1}\|u_{m+1}\|_{L^2}^2 + \sum_{i=0}^{k-1}(c_{m_0+1+2i} - (B+1))\|u_{m_0+1+2i}\|_{L^2}^2 \\ &\quad + \sum_{i=0}^{k}(c_{m_0+2i} - C)\|u_{m_0+2i}\|_{L^2}^2 - (B+1)\|u_{m_0-1}\| \\ &\geq c_{m+1}\|u_{m+1}\|_{L^2}^2 - (B+1)\|u_{m_0-1}\|_{L^2}^2. \end{aligned}$$

Hence for any such $m = m_0 + 2k$ we get that

$$a_{m+1} \geq a_{m_0-1} - (B+1)\|u_{m_0-1}\|_{L^2}^2.$$

By the definition of a_{m_0-1} and (5.28) we have that $a_{m_0-1} \geq c_{m_0-1}\|u_{m_0-1}\|_{L^2}^2$, and hence we finally arrive at

$$a_{m+1} \geq (c_{m_0-1} - (B+1))\|u_{m_0-1}\|_{L^2}^2.$$

Assume we defined m_0 before so that $c_{m_0-1} \geq B+1$ as well. Observe that $a_{m+1} \to 0$ as $m \to \infty$ by Lemma 5.30. Hence we get that $\|u_{m_0-1}\|_{L^2}^2 = 0$ and thus $u_{m_0-1} = 0$ for all such large enough m_0. This proves the theorem in the case $n \neq 2$.

Finally, let's discuss the modification needed in the case $n = 2$. In this case we instead use the second equality in (5.28) and hence proceeding as above arrive at that for sufficiently large m

$$d_m a_{m+1} \geq a_{m-1} + r_m,$$

where we've used that $d_m \geq d_{m+1}$ and that all $d_i \geq 1$ for $i \geq 1$. Multiplying through by d_{m-2}, applying the same inequality with "m" replaced by "$m-2$" on the right-hand side, and then repeating recursively gives

$$(d_m \cdot \ldots \cdot d_{m_0+2})a_{m+1} = a_{m_0-1} + r_{m_0} + (d_{m_0+2})r_{m_0+2} + \ldots + (d_{m-2} \cdot \ldots \cdot d_{m_0+2})r_m$$

for any pair of $m, m_0 > \deg f$ such that $m = m_0 + 2k$ for some integer k. Since the $d_i \geq 1$, we get the inequality

$$\left(\prod_{i=0}^{k-1} d_{m-2i}\right) a_{m+1} \geq a_{m_0-1} + r_{m_0} + r_{m_0+2} + \ldots + r_m.$$

Again by the definition of a_{m_0-1} and (5.28) we have that $a_{m_0} \geq ({}^{c_{m_0-1}}/_{d_{m_0-1}})\|u_{m_0-1}\|_{L^2}^2$ and so

$$\left(\prod_{i=0}^{k-1} d_{m-2i}\right) a_{m+1} \geq \left(\frac{c_{m_0-1}}{d_{m_0-1}} - (B+1)\right)\|u_{m_0-1}\|_{L^2}^2.$$

Since the $d_m \to 1$ as $m \to \infty$, we can assume that we defined m_0 before so that ${}^{c_{m_0-1}}/_{d_{m_0-1}} \geq B+1$ as well. Furthermore, $\prod_{m=1}^{\infty} d_m$ converges by the infinite product criteria since $\sum_{m=1}^{\infty}(d_m - 1) < \infty$ and so the coefficient on the left-hand side is bounded by some fixed constant. Hence the theorem follows again from the fact that $a_{m+1} \to 0$ as $m \to \infty$.

■

Theorem 5.23 has one disadvantage. Though it tells us that f being of finite degree implies that the solution u is also of finite degree, it gives no information about the degree of u itself. This can remedied if we assume an additional condition on the metric g and connection $\nabla^{\mathcal{E}}$. For any index $m \geq 0$, we call elements of $\ker \mathbb{X}_+|_{\Omega_m}$ **twisted conformal Killing tensors (CKT's)** of degree m. We say that there are no nontrivial CKTs that decay to order ${}^{(n+1)}/_2$ or faster at infinity if all CKTs of order $m \geq 1$ that are in $\rho^\alpha C^\infty\left({}^0S^*\overline{M}; \pi^*\mathcal{E}\right)$ for $\alpha \geq {}^{(n+1)}/_2$ are identically zero.

Theorem 5.36. *Assume that the sectional curvatures of g are negative. Suppose also that there are no nontrivial CKTs that decay to order ${}^{(n+1)}/_2$ or faster at infinity. If $u \in \rho^\alpha C^\infty\left({}^0S^*\overline{M}; \pi^*\mathcal{E}\right)$, where $\alpha \geq {}^{(n+1)}/_2$, solves*

$$\mathbb{X}u + \Phi u = f$$

for some $f \in C^\infty\left({}^0S^*\overline{M}; \pi^*\mathcal{E}\right)$ of finite degree m, then u is of degree $\max\{m-1, 0\}$.

Proof: We prove this by contradiction: suppose not! Suppose that $l = \deg u \geq m$. Then comparing the Fourier modes of order $l+1$ of both sides of the above equation gives

$$\mathbb{X}_+ u_l = 0.$$

Since we assumed that there are no nontrivial CKTs that decay to order ${}^{(n+1)}/_2$ or faster at infinity, this implies that $u_l \equiv 0$ and hence contradicts the assumption that the degree of u is l.

■

5.3 Regularity of Solutions to the Transport Equation

Before we prove the main result of our paper, we need to establish the regularity of solutions to the transport equation. We're interested in solutions to the transport equation that extend smoothly to ${}^bS^*\overline{M}$ and ${}^0S^*\overline{M}$. Here we use the material that we introduced in Section 2.4 up to (2.9) there and the two sentences after.

Proposition 5.37. *Assume that (M,g) is nontrapping. Suppose that $\Phi \in \rho^\infty C^\infty\left(\overline{M}; \mathrm{End}_{\mathrm{Sk}}\mathcal{E}\right)$ and that the connection symbols of $\nabla^{\mathcal{E}}$ are in $\rho^\infty C^\infty\left(\overline{M};\mathbb{R}\right)$ in any boundary coordinates and frame. Suppose also that we have $\phi \in \rho^\infty C^\infty\left(\overline{M};\mathcal{E}\right)$ and $A \in \rho^\infty C^\infty\left({}^bT\overline{M};\mathcal{E}\right)$ such that for every fixed $x \in \overline{M}$, $A_x : {}^bT_xM \to \mathcal{E}_x$ is a linear map.*

Then there exists a unique solution $u \in C^\infty\left({}^bS^*\overline{M}; \pi_b^*\mathcal{E}\right)$ to

$$\mathbb{X}u + \Phi u = \phi + A \qquad \text{on } SM, \tag{5.38}$$

with $u|_{\partial_- {}^bS^*\overline{M}} = h$ for any given boundary data $h \in C^\infty\left(\partial_- {}^bS^*\overline{M}; \pi_b^*\mathcal{E}|_{\partial_- {}^bS^*\overline{M}}\right)$. If $h \equiv 0$ and $u|_{\partial_+ S^*\overline{M}} \equiv 0$, then $u|_{SM}$ extends smoothly to be an element of $\rho^\infty C^\infty\left({}^0S^*\overline{M};\pi_0^*\mathcal{E}\right)$.

Proof: Recall that $d = \operatorname{rank}\mathcal{E}$. From Lemma 2.1 in [12] we have that $X = \rho\overline{X}$ for some smooth vector field $\overline{X}$ over ${}^bS^*\overline{M}$ that is transverse to $\partial^bS^*\overline{M}$. Let $(\rho, y^\mu) = (x^i)$ be boundary coordinates of $\overline{M}$, (b_k) a frame for $\mathcal{E}$ over their domain, and consider the coordinates $v^i\partial/\partial x^i \mapsto (x^i, v^i)$ of TM. Let ${}^{\mathcal{E}}\Gamma^k_{ij}$ denote the connection symbols of $\nabla^{\mathcal{E}}$ in these coordinates and frame. Then observe that the components of (5.38) with respect to $(\pi^* b_k)$ are given by

$$Xu^k + {}^{\mathcal{E}}\Gamma^k_{ij}v^iu^j + \Phi^k_ju^j = \phi^k + A^k_iv^i \tag{5.39}$$

where $k = 1, \ldots, d$. Hence over SM

$$\overline{X}u^k + \rho^{-1}{}^{\mathcal{E}}\Gamma^k_{ij}v^iu^j + \rho^{-1}\Phi^k_ju^j = \rho^{-1}\phi^k + \rho^{-1}A^k_iv^i. \tag{5.40}$$

Take the coordinates $\eta_0 {d\rho}/{\rho} + \eta_\lambda dy^\lambda \mapsto (x^i, \eta_i)$ of ${}^bT^*\overline{M}$ and observe that the canonical identification $\sharp \circ (F^*)^{-1} : {}^bT^*\overline{M}\big|_M \to TM$ is given by

$$v^i = g^{ii'}\begin{cases} \eta_0/\rho & \text{if} \quad i' = 0 \\ \eta_{i'} & \text{if} \quad i' > 0 \end{cases}.$$

We denote the right-hand side by $g^{ii'}\{\eta_{i'}\}$. Pulling the above equation (5.40) to ${}^bS^*\overline{M}$ gives that the components of (5.38) with respect to $(\pi_b^*b_k)$ satisfy

$$\overline{X}u^k + \rho^{-1}{}^{\mathcal{E}}\Gamma^k_{ij}g^{ii'}\{\eta_{i'}\}u^j + \rho^{-1}\Phi^k_ju^j = \rho^{-1}\phi^k + \rho^{-1}A^k_ig^{ii'}\{\eta_{i'}\}. \tag{5.41}$$

We remind the reader that each g^{ij} is ρ^2 times something smooth on $\overline{M}$. Since we assumed that g is nontrapping, it follows from Lemma 2.3 in [12] that any maximal integral curve[7] σ of $\overline{X}$ is of the form $\sigma : [a,b] \to {}^bS^*\overline{M}$ where a and b are finite and $\sigma(a) \in \partial_-S^*\overline{M}$, $\sigma(b) \in \partial_+S^*\overline{M}$. Hence, (5.41) can be viewed as ordinary differential equation (ODEs) along

[7] i.e. integral curve whose interval domain cannot be extended.

such curves σ. Hence, noting that $\overline{X}$ is nonvanishing on ${}^bS^*\overline{M}$, using the theory of flows and the existence, uniqueness, and smooth dependence on initial condition of linear ODEs (see [5] and [21]), it's not hard to see from here that indeed a unique smooth solution u exists to (5.38) satisfying the given boundary data.

Next suppose that $h \equiv 0$ and that $u|_{\partial_+ S^*\overline{M}} \equiv 0$. Pick any point $x_0 \in \partial M$ contained in our coordinates (x^i). We will show that for some neighborhood W of x_0 in $\overline{M}$, $u|_{\pi^{-1}[W]}$ extends to an element in $\rho^\infty C^\infty\big(\pi_0^{-1}[W]; \pi_0^*\mathcal{E}\big)$ (by setting $u \equiv 0$ on $\partial^0 S^*\overline{M}$) from which the theorem will follow. It's not hard to see that this claim will follow if we show that for any finite collection $V_1, \ldots, V_m \in C^\infty\big({}^0S^*\overline{M}; T^0S^*\overline{M}\big)$ of smooth vector fields over ${}^0S^*\overline{M}$,

$$\text{each } V_1 \ldots V_m u^k \text{ over } SM \text{ is in } \rho^\infty L^\infty\big(\pi^{-1}[W]; \pi^{-1}\mathcal{E}\big). \tag{5.42}$$

We will do this by induction on m, where our approach will be to study the growth of the solution to (5.39) and its derivatives by writing that equation as an ODE along integral curves of X.

We begin by recalling a geometric fact about AH spaces. Let $\varphi : SM \times \mathbb{R} \to SM$ denote the flow of X. For any point $\zeta = \eta_0{}^{d\rho}/_{\rho} + \eta_\lambda dy^\lambda \in {}^bS^*M$ we write its identified point on SM as $z \in SM$. By Lemma 2.3 in [12] and its proof there exists a constant $C > 0$ independent of $\zeta \cong z$ such that if we write $\rho(t) = \rho \circ \varphi_z(t)$,

1. if $\eta_0 \geq 0$ then $\lim_{t\to-\infty} \varphi_z(t) \in \partial_- {}^bS^*\overline{M}$ (in ${}^bS^*\overline{M}$), $\rho(t) \leq Ce^t$, $\rho(t)$ is increasing, and the image $\varphi_z(-\infty, 0]$ is contained in a compact subset of ${}^bS^*\overline{M}$,
2. if $\eta_0 < 0$ then $\lim_{t\to+\infty} \varphi_z(t) \in \partial_+ {}^bS^*\overline{M}$ (in ${}^bS^*\overline{M}$), $\rho(t) \leq Ce^{-t}$, $\rho(t)$ is decreasing, and the image $\varphi_z[0, \infty)$ is contained in a compact subset of ${}^bS^*\overline{M}$.

By the same lemma it follows that there exist neighborhoods W and K of x_0 in $\overline{M}$ in our coordinates such that $W \subseteq K$ and that for any $z \in \pi^{-1}[W]$, φ_z will always be contained in $\pi^{-1}[K]$ for $t > 0$ if $\eta_0 \geq 0$ or $t < 0$ if $\eta_0 < 0$. For convenience, we assume that K is compact in $\overline{M}$.

We begin with the case $m = 0$ (i.e. there are no V_i's). Fix any $N \in \mathbb{R}$. Fix a point $\zeta' = \eta_0'{}^{d\rho}/_{\rho} + \eta_\lambda' dy^\lambda \in \pi_b^{-1}[W]$ identified with $z' \in SM$. Suppose that $\eta_0' \geq 0$ since the proof below is essentially the same for the case $\eta_0' < 0$. Hence we have that $\varphi_{z'}(-\infty, 0]$ is contained $\pi^{-1}[K]$ and $\lim_{t\to-\infty} \varphi_{z'}(t) \in \partial_- {}^bS^*\overline{M}$. Because we're working on the sphere bundle and $g = \rho^{-2}\overline{g}$ for a smooth metric $\overline{g}$, there exists a constant $C' > 0$ such that the magnitude of v^i in (5.39) are less than C' on $\pi^{-1}[K]$. This way, setting B and b to be the smooth $d \times d$ matrix and $d \times 1$ column vector respectively given by

$$B_i^k = \big({}^{\mathcal{E}}\Gamma_{ij}^k v^i + \Phi_j^k\big) \qquad \text{and} \qquad b^k = \phi^k + A_i^k v^i, \tag{5.43}$$

we have by (5.39) that $u^k \circ \varphi_{z'}$ satisfies the ODE

$$\frac{du^k}{dt} + \big(B_i^k \circ \varphi_{z'}\big) u^i = b^k \circ \varphi_{z'}. \tag{5.44}$$

For any matrix M or column vector w, let $|M|$ and $|w|$ denote the norms $\sum_{ik}\big|M_i^k\big|$ and $\sum_i |w^i|$. Observe that both $|B|, |b| \leq C''\rho^N$ on $\pi^{-1}[K]$ for some constant $C'' > 0$. Our boundary conditions imply that $u^k(t) \to 0$ as $t \to -\infty$, and so by the fundamental theorem of calculus and the triangle inequality

$$|u(t)| \leq \int_{-\infty}^{t} |B(s)||u(s)|ds + \int_{-\infty}^{t} |b(s)|ds \tag{5.45}$$

where u denotes the column vector whose components are u^i. We note that these integrals converge because both $|B|, |b| \leq C'\rho^N$, $\rho(t) \leq Ce^t$, and $|u|$ is bounded on the precompact $\varphi_\zeta(-\infty, 0]$. Next, we employ the standard trick in ODEs of defining $R : \mathbb{R} \to \mathbb{R}$ to be the first integral on the right-hand side. Whenever $|B(t)| \neq 0$,

$$\frac{1}{|B(t)|} R'(t) \leq R(t) + \int_{-\infty}^{t} |b(s)|ds$$

and so

$$R'(t) \leq |B(t)|R(t) + |B(t)| \int_{-\infty}^{t} |b(s)|ds.$$

Observe that this inequality also holds when $|B(t)| = 0$ and hence for all t. This is a separable equation. In particular, if we take $|B(t)|R(t)$ to the left-hand side, multiply through by $\exp\left[-\int_{-\infty}^{t} |B(s)|ds\right]$, integrate from $t = -\infty$ to $t = \tau$ (using that $R(-\infty) = 0$), and finally divide through by $\exp\left[-\int_{-\infty}^{\tau} |B(s)|ds\right]$ we get that

$$R(\tau) \leq e^{\int_{-\infty}^{\tau} |B(s)|ds} \int_{-\infty}^{\tau} e^{-\int_a^t |B(s)|ds} |B(t)| \int_{-\infty}^{t} |b(s)|dsdt.$$

We're interested in $\tau = 0$ since that's when $\varphi_{z'}$ reaches our point of interest z'. We now perform estimates. Trivially $\exp\left[-\int_a^t |B(s)|ds\right] \leq 1$. Next,

$$\int_{-\infty}^{0} |B(s)|ds \leq C''C^N \int_{-\infty}^{0} e^{Nt}dt \leq C''C^N.$$

We also have that $|B(t)| \leq C''C^N\rho^{N-1}(0)\rho(t)$ and $|b(s)| \leq C''C^N\rho^{N-1}(0)\rho(s)$ since ρ is increasing along $\varphi_{z'}$. Hence making a similar estimate for $\int_{-\infty}^{0} |B(t)| \int_{-\infty}^{t} |b(s)|dsdt$ gives us that

$$R(0) \leq e^{C''C^N} \left(C''C^N\right)^2 \rho^{2N-2}(z').$$

Assume for convenience that $N \geq 1$. Then plugging this inequality and the estimate $\int_{-\infty}^{t} |b(s)|ds \leq C''C^N\rho^{N-1}(t)$ into (5.45) finally gives us that

$$|u(z')| \leq C'''\rho^{2N-2}(z') \tag{5.46}$$

for all $z' \in \pi^{-1}[W]$ for some constant $C''' > 0$ independent of u and z'. Since N can be made arbitrary large, this proves the case $m = 0$.

Next we do the case $m = 1$, after which it should be clear how the induction works for higher m. Fix some $V \in C^\infty\left({}^0S^*\overline{M}; T^0S^*\overline{M}\right)$. Let's see what this vector field looks like when pushed to SM over the interior. Take the coordinates $\overline{\eta}_i{}^{dx^i}/\rho \mapsto (x^i, \overline{\eta}_i)$ of ${}^0T^*\overline{M}$ (note the bars to distinguish these from our coordinates of ${}^bT^*\overline{M}$ above). Writing $V = V^i\partial/\partial x^i + V_i\partial/\partial\overline{\eta}_i$ and observing the canonical identification $v^i = \rho^{-1}g^{ij}\overline{\eta}_j$, a quick computation gives that

$$V \text{ over } SM = V^i\frac{\partial}{\partial x^i} + \Big(-\rho^{-1}v^iV^0 + \partial_{x^k}\big(g^{ij}\big)g_{jj'}v^{j'}V^k\Big)\frac{\partial}{\partial v^i} + \sum_{i=1}^{n+1}\rho^{-1}g^{ii}V_i\frac{\partial}{\partial v^i}. \tag{5.47}$$

We point out that the V^i and V_i here are bounded over $\pi^{-1}[K]$ due the compactness of $\pi_0^{-1}[K]$.

Starting similarly as we did in the proof of the "$m = 0$" case above, by (5.44) we have that (here $Vu^k = V\big(u^k \circ \varphi\big)$)

$$\frac{d\big(Vu^k\big)}{dt} + V\big(B_i^k \circ \varphi\big)(z', t)u^i + \big(B_i^k \circ \varphi_{z'}\big)Vu^i = V\big(b^k \circ \varphi\big)(z', t). \tag{5.48}$$

Let's study the growth rate of z-partials of the terms $B_i^k \circ \varphi$ and $b^k \circ \varphi$ near the boundary. As we discussed in the proof of Lemma 5.2 above, X over SM extends to a smooth vector field "X_0" over ${}^0S^*\overline{M}$ and hence has a smooth flow φ_0 over ${}^0S^*\overline{M}$. Similarly to what is mentioned at the end of the proof of Lemma 3.13 in [12], the compactness of ${}^0S^*\overline{M}$ and Grönwall's inequality imply that for any $V_1, \ldots, V_j \in C^\infty\big({}^0S^*\overline{M}; T^0S^*\overline{M}\big)$,

$$\Big|V_1 \ldots V_j(\varphi_0)_t^k\Big| \le C_0e^{k_0|t|}$$

for $\big(\overline{\zeta}, t\big)$ such that both $\big(\overline{\zeta}, t\big), \varphi_0\big(\overline{\zeta}, t\big) \in \pi_0^{-1}[K]$ where $C_0 > 0$ and $k_0 > 0$ are constants *dependent* on $V_1, \ldots, V_j$ and the $(\varphi_0)_t^k$'s are the components of $(\varphi_0)_t$ with respect to our coordinates $(x^i, \overline{\eta}_i)$ of ${}^0T^*\overline{M}$. Trivially $\varphi = (\varphi_0)$ over SM (up to identification) and so, it's not hard to see by (5.43) and (5.47) that

$$\text{both } \big|V_1 \ldots V_j\big(B_i^k \circ \varphi\big)(z, t)\big| \text{ and } \big|V_1 \ldots V_j\big(b^k \circ \varphi\big)(z, t)\big| \text{ are } \le \widetilde{C}\rho^N(z)e^{\widetilde{k}_0|t|}$$

for (z, t) such that both $(z, t), \varphi(z, t) \in \pi^{-1}[K]$ where $\widetilde{C}, \widetilde{k}_0 > 0$ are constants dependent on $V_1, \ldots, V_j$. By similar reasoning, (5.46) and the inequality $\rho(t) \le Ce^t$ show that

$$V\big(u^k \circ \varphi\big)(z', t) \to 0 \qquad \text{as } t \to -\infty.$$

Applying this to (5.48), knowing (5.46) one can proceed similarly as before to come to the conclusion that $|Vu| \le \widetilde{C}'\rho^{2N-2}$ on $\pi^{-1}[W]$ for some constant $\widetilde{C}' > 0$ dependent on V where Vu is the column vector with entries Vu^i. The induction process proceeds similarly for higher m.

■

5.4 Proof of Theorem 2.6

The first step is to provide a formulation of (2.3) as a single transport equation of an endomorphism field over SM. To begin, we define a natural connection "$\nabla^{\pi^*\mathrm{En}}$" on endomorphism fields as follows. For any $U \in C^\infty(SM; \operatorname{End} \pi^*\mathcal{E})$ and any $\omega \in T_v SM$, we define $\nabla^{\pi^*\mathrm{En}}_\omega U$ to be the unique element of $\operatorname{End} \pi^*\mathcal{E}_x$ satisfying that

$$\big(\nabla^{\pi^*\mathrm{En}}_\omega U\big)h := \big[\nabla^{\pi^*\mathcal{E}}_\omega, U\big]h,$$

for any $h \in C^\infty(SM; \operatorname{End} \pi^*\mathcal{E})$. To see what this looks like in coordinates, take coordinates (x^i) of M, a frame (b_i) for $\mathcal{E}$ over their domain, and consider the coordinates $v^i \partial/\partial x^i \mapsto (x^i, v^i)$ of TM. Let ${}^{\mathcal{E}}\Gamma^k_{ij}$ denote the connection symbols of $\nabla^{\mathcal{E}}$ in these coordinates and frame. A quick computation shows that

$$\nabla^{\pi^*\mathrm{En}}_\omega U = \omega U + \big({}^{\mathcal{E}}\Gamma\big)U - U\big({}^{\mathcal{E}}\Gamma\big), \tag{5.49}$$

where on the right-hand side U is thought of as a matrix in the basis $(\pi^* b_i)$, ωU denotes applying ω to the entries of U, and ${}^{\mathcal{E}}\Gamma$ represents the matrix with entry ${}^{\mathcal{E}}\Gamma^k_{ij} v^i$ in the k^{th} row and j^{th} column (recall that "v" is where ω is based at). We point out that this is considered a natural connection on endomorphism fields because it satisfies the product rule $\nabla^{\pi^*\mathcal{E}}_\omega(Uh) = \big(\nabla^{\pi^*\mathrm{En}}_\omega U\big)h + U\nabla^{\pi^*\mathcal{E}}_\omega h$. We define the operator $\mathbb{X} = \nabla^{\pi^*\mathrm{En}}_X$, using context to differentiate it from our other operators also denote by "$\mathbb{X}$."

By looking in local coordinates it's easy to see that $\widetilde{\nabla}^{\mathcal{E}} = \nabla^{\mathcal{E}} + A$ (and hence $\widetilde{\nabla}^{\pi^*\mathcal{E}} = \nabla^{\pi^*\mathcal{E}} + A$ up to identification) where $A \in C^\infty\big(T\overline{M}; \operatorname{End} \mathcal{E}\big)$ is a skew-Hermitian connection 1-form. It's also easy to see that in any boundary coordinates and frame the entries of A are in $\rho^\infty C^\infty\big(\overline{M}; \mathbb{R}\big)$. Consider solutions U and $\widetilde{U}$ to the following transport equations on SM:

$$\begin{cases} \mathbb{X}U + \Phi U = 0, & U|_{\partial_- {}^b S^*\overline{M}} = \mathrm{id}, \\ \mathbb{X}\widetilde{U} + A\widetilde{U} + \widetilde{\Phi}\widetilde{U} = 0, & \widetilde{U}\Big|_{\partial_- {}^b S^*\overline{M}} = \mathrm{id}, \end{cases} \tag{5.50}$$

which exist by Proposition 5.37 (using a different connection and Higgs field), where we note that in the second equation "$\widetilde{\mathbb{X}}\widetilde{U} - \widetilde{U}A$" can be thought of as "$\nabla_X \widetilde{U}$" for some connection "$\nabla$."

We demonstrate the usefulness of U and $\widetilde{U}$. Suppose that $\gamma : (-\infty, \infty) \to M$ is a complete unit-speed geodesic and that $u : (-\infty, \infty) \to \mathcal{E}$ is the smooth solution along γ to the initial value problem

$$\nabla^{\mathcal{E}}_{\dot\gamma(t)} u(t) = 0, \qquad \lim_{t \to -\infty} u(\gamma(t)) = e, \tag{5.51}$$

where e is an element in $\mathcal{E}_{x_0}$ where x_0 is the limit of $\gamma(t)$ in $\overline{M}$ as $t \to -\infty$. We point out that such a u exists by Lemma 2.5 with $\Phi = 0$. Let $\sigma : (-\infty, \infty) \to SM$ be the integral curve of X satisfying $\gamma = \pi \circ \sigma$ and let $\hat{u}$ denote u lifted to σ via canonical identifications of $\mathcal{E}$ and $\pi^*\mathcal{E}$. It's easy to see that $\nabla^{\pi^*\mathcal{E}}_{\dot\sigma}\hat{u} = 0$. Then

$$\nabla^{\mathcal{E}}_{\dot\gamma}((U \circ \sigma)u) + \Phi((U \circ \sigma)u) \cong \nabla^{\pi^*\mathcal{E}}_{\dot\sigma}(U\hat{u}) + \Phi(U\hat{u}) = \big(\nabla^{\pi^*\mathrm{En}}_X U\big)\hat{u} + U\nabla^{\pi^*\mathcal{E}}_{\dot\sigma}\hat{u} + \Phi(U\hat{u}) = 0,$$

$$\text{with}\quad \lim_{t\to-\infty}(U\circ\sigma)u(t)=e,$$

and similarly

$$\widetilde{\nabla}^{\mathcal{E}}_{\dot\gamma}\Big(\Big(\widetilde{U}\circ\sigma\Big)\hat u\Big)+\widetilde{\Phi}\Big(\Big(\widetilde{U}\circ\sigma\Big)\hat u\Big)\cong\widetilde{\nabla}^{\pi^*\mathcal{E}}_{\dot\sigma}\Big(\widetilde{U}\hat u\Big)+\widetilde{\Phi}\Big(\widetilde{U}\hat u\Big)=\nabla^{\pi^*\mathcal{E}}_{\dot\sigma}\Big(\widetilde{U}\hat u\Big)+A\Big(\widetilde{U}\hat u\Big)+\widetilde{\Phi}\Big(\widetilde{U}\hat u\Big)$$

$$=\Big(\nabla^{\pi^*\mathrm{En}}_X\widetilde{U}\Big)\hat u+\widetilde{U}\nabla^{\pi^*\mathcal{E}}_{\dot\sigma}\hat u+A\Big(\widetilde{U}\hat u\Big)+\widetilde{\Phi}(U\hat u)=0,$$

$$\text{with}\quad \lim_{t\to-\infty}\Big(\widetilde{U}\circ\sigma\Big)u(t)=e.$$

Hence $(U\circ\sigma)u(t)$ and $\Big(\widetilde{U}\circ\sigma\Big)u(t)$ are solutions to (2.3) and (2.3) with Φ and $\nabla^{\mathcal{E}}$ replaced by $\widetilde{\Phi}$ and $\widetilde{\nabla}^{\mathcal{E}}$ respectively. By our assumption the data (2.4) is the same for both and so

$$\lim_{t\to\infty}(U\circ\sigma)u(t)=\lim_{t\to\infty}\Big(\widetilde{U}\circ\sigma\Big)u(t).$$

Since parallel transport such as (5.51) above is an isomorphism between fibers, this implies that $U=\widetilde{U}$ on $\partial^b S^*\overline{M}$.

Intuitively speaking, we've demonstrated that knowing the parallel transport (5.51), the endomorphism fields U and $\widetilde{U}$ encode the transform that takes all possible (γ,e) to the data (2.4) for $\nabla^{\mathcal{E}},\Phi$ and $\widetilde{\nabla}^{\mathcal{E}},\widetilde{\Phi}$ respectively. Furthermore, the assumption that the two transforms are equal gives us that U and $\widetilde{U}$ are equal on the boundary. Hence we've reformulated our problem to showing that $U=\widetilde{U}$ on $\partial^b S^*M$ implies the gauge equivalence stated in the theorem.

Guided by the observation (2.8) in the introduction, we next study the behavior of $U\widetilde{U}^{-1}$ over the interior SM. We note that both U and $\widetilde{U}$ are invertible because (5.50) can be seen as ordinary differential equations along integral curves of X (c.f. (1.8) in Chapter 3 of [5]). A quick computation using (5.49) shows that $\mathbb{X}(W_1W_2)=\mathbb{X}(W_1)W_2+W_1\mathbb{X}(W_2)$ and that $\mathbb{X}(\mathrm{id})=0$. Hence by (5.50),

$$\mathbb{X}\Big(\widetilde{U}^{-1}\Big)=-\widetilde{U}^{-1}\widetilde{\mathbb{X}}\Big(\widetilde{U}\Big)\widetilde{U}^{-1}=\widetilde{U}^{-1}\Big(A\widetilde{U}+\widetilde{\Phi}\widetilde{U}\Big)\widetilde{U}^{-1}=\widetilde{U}^{-1}A+\widetilde{U}^{-1}\widetilde{\Phi}.$$

Next, we have that

$$\mathbb{X}\Big(U\widetilde{U}^{-1}\Big)=(-\Phi U)\widetilde{U}^{-1}+U\Big(\widetilde{U}^{-1}A+\widetilde{U}^{-1}\widetilde{\Phi}\Big),$$

and so we finally arrive at that $Q=U\widetilde{U}^{-1}$ satisfies

$$\mathbb{X}Q+\Phi Q-QA-Q\widetilde{\Phi}=0. \tag{5.52}$$

To apply our finite degree theorems from Section 5.2 above, we need our solution to vanish at "infinity." Hence we instead consider $W=Q-\mathrm{id}$ which satisfies

$$\mathbb{X}W+\Phi W-WA-W\widetilde{\Phi}=-\Phi+A+\widetilde{\Phi}.$$

As before, $\mathbb{X}W-WA$ can be thought of as $\nabla_X W$ for some connection "∇," and $\Phi W-W\widetilde{\Phi}$ can be thought of an element of $\rho^\infty C^\infty\big(\overline{M};\mathrm{End}\,\mathrm{End}\,\mathcal{E}\big)$ applied to W. Hence, by Proposition

5.37 we have that W and hence Q are in $\rho^\infty C^\infty\big({}^0S^*\overline{M}, \pi_0^*\mathcal{E}\big)$. In the coordinates and frame used in (5.49), above any fixed point $x \in M$ the entries of the right-hand side with respect to the basis $(\pi^* b_i)$ are restrictions of homogeneous polynomials of order zero and one in the variable v. By the theory of spherical harmonics these are elements of (Fourier) degree zero and one respectively. Hence by Theorem 5.36 we have that W and hence Q are of degree zero. In other words, we get that $Q \in C^\infty\big(\overline{M}; \operatorname{End}\mathcal{E}\big)$ (i.e. up to identification).

As the final step, let's show that this Q is the gauge that we wanted. Adopting the coordinates and frames in (5.49), we have that

$$\mathbb{X}Q = v(Q) + \big({}^{\mathcal{E}}\Gamma\big)Q - Q\big({}^{\mathcal{E}}\Gamma\big).$$

Hence, using the theory of spherical harmonics, equating the zeroth and first Fourier modes in (5.52) gives

$$\Phi Q - Q\widetilde{\Phi} = 0 \qquad \text{and} \qquad \mathbb{X}Q - QA = 0.$$

The first equation gives $\widetilde{\Phi} = Q^{-1}\Phi Q$ over M and hence over $\overline{M}$. The second equation gives

$$\begin{aligned} A = Q^{-1}\mathbb{X}Q = Q^{-1}\big[v(Q) + \big({}^{\mathcal{E}}\Gamma\big)Q - Q\big({}^{\mathcal{E}}\Gamma\big)\big], \\ \implies \qquad A + \big({}^{\mathcal{E}}\Gamma\big) = Q^{-1}v(Q) + Q^{-1}\big({}^{\mathcal{E}}\Gamma\big)Q, \end{aligned}$$

Now, take any section $u \in C^\infty(M; \mathcal{E}|_M)$ and write it as a column vector with respect to the basis (b_i). We have that (here $v(u)$ denotes applying v to every entry of u)

$$\begin{aligned} \widetilde{\nabla}^{\mathcal{E}}_v u &= v(u) + \big[A + \big({}^{\mathcal{E}}\Gamma\big)\big]u && \text{eq. for } \widetilde{\nabla}^{\mathcal{E}}_v, \\ &= v\big(Q^{-1}Qu\big) + Q^{-1}v(Q)u + Q^{-1}\big({}^{\mathcal{E}}\Gamma\big)Qu && QQ^{-1} = \text{id and plug in eq. above}, \\ &= v\big(Q^{-1}Qu\big) - v\big(Q^{-1}\big)Qu + Q^{-1}\big({}^{\mathcal{E}}\Gamma\big)Qu && Q^{-1}v(Q) = -v\big(Q^{-1}\big)Q \text{ (prod. rule)}, \\ &= Q^{-1}v(Qu) + Q^{-1}\big({}^{\mathcal{E}}\Gamma\big)Qu && \text{prod. rule}, \\ &= Q^{-1}\nabla^{\mathcal{E}}_v(Qu) && \text{eq. for } \nabla^{\mathcal{E}}_v. \end{aligned}$$

Again this relation extends to ∂M by continuity and hence the theorem is proved.

■

6 Injectivity over Higgs Fields

In this section we prove Corollary 2.8. Recall that $d = \operatorname{rank}\mathcal{E}$. By Theorem 2.6 we know that there exists a $Q \in C^\infty\big(\overline{M}; \operatorname{End}\mathcal{E}\big)$ such that $Q|_{\partial\overline{M}} = \text{id}$ and satisfies (2.7) with $\widetilde{\nabla} = \nabla$. Hence, we simply need to show that $Q \equiv \text{id}$ everywhere. Take coordinates (x^i) of M, a frame (b_i) for $\mathcal{E}$ over their domain, and consider the coordinates $v^i\partial/\partial x^i \mapsto (x^i, v^i)$ of TM. Let ${}^{\mathcal{E}}\Gamma^k_{ij}$ denote the connection symbols of $\nabla^{\mathcal{E}}$ in these coordinates and frame. Since the curvature of $\nabla^{\mathcal{E}}$ is zero, by Proposition 1.2 of Appendix C in Volume II of [47], we may suppose that the (b_i) were chosen so that all of the ${}^{\mathcal{E}}\Gamma^k_{ij} \equiv 0$. Let us represent Q as a $d \times d$ matrix in the basis (b_i). Similarly we represent any section $u \in C^\infty\big(\overline{M}; \mathcal{E}\big)$ as a $d \times 1$ column vector in this basis. Then for any section $u = u^j b_j \cong [u^1, \ldots, u^d]$ whose component functions u^j are constant, we have that for any $v \in TM$ in our coordinates

$$\nabla^{\mathcal{E}}_v u = 0,$$

$$\widetilde{\nabla}^{\mathcal{E}}_v u = Q^{-1}\nabla^{\mathcal{E}}_v (Qu) = Q^{-1} v(Q) u.$$

where $v(Q)$ denotes applying v to the entries of Q. Since $\widetilde{\nabla}^{\mathcal{E}}_v u = \nabla^{\mathcal{E}}_v u$ by assumption and the above is true for all such u, we get that $v(Q) \equiv 0$. Hence Q is locally constant, and thus indeed equal to "id" everywhere.

■

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