ON ABSTRACT STRICHARTZ ESTIMATES AND THE STRAUSS CONJECTURE FOR NONTRAPPING OBSTACLES

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1. **Introduction.**

The purpose of this paper is to show how local energy decay estimates for certain linear wave equations involving compact perturbations of the standard Laplacian lead to optimal global existence theorems for the corresponding small amplitude nonlinear wave equations with power nonlinearities. To achieve this goal, at least for spatial dimensions $n = 3$ and 4, we shall show how the aforementioned linear decay estimates can be combined with "abstract Strichart" estimates for the free wave equation to prove corresponding estimates for the perturbed wave equation when $n \geq 3$. As we shall see, we are only partially successful in the latter endeavor when the dimension is equal to two, and therefore, at present, our applications to nonlinear wave equations in this case are limited.

Let us start by describing the local energy decay assumption that we shall make throughout. We shall consider wave equations on the exterior domain $\Omega \subset \mathbb{R}^n$ of a compact obstacle:

$$\begin{cases} (\partial_t^2 - \Delta_{\mathrm{g}})u(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(0,\,\cdot\,) = f \\ \partial_t u(0,\,\cdot\,) = g \\ (Bu)(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.1}$$

where for simplicity we take B to either be the identity operator (Dirichlet-wave equation) or the inward pointing normal derivative ∂_ν (Neumann-wave equation). We shall also assume throughout that the spatial dimension satisfies $n \geq 2$.

The operator Δ_{g} is the Laplace-Beltrami operator associated with a smooth, time independent Riemannian metric $\mathbf{g}_{jk}(x)$ which we assume equals the Euclidean metric δ_{jk} for $|x| \geq R$, some R. The set Ω is assumed to be either all of $\mathbb{R}^n$, or else $\Omega = \mathbb{R}^n \backslash \mathcal{K}$ where $\mathcal{K}$ is a compact subset of $|x| < R$ with smooth boundary.

We can now state the main assumption that we shall make.

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Hypothesis B. *Fix the boundary operator B and the exterior domain $\Omega \subset \mathbb{R}^n$ as above. We then assume that given $R_0 > 0$*

$$\int_0^\infty \Big(\|u(t,\,\cdot\,)\|^2_{H^1(|x|<R_0)} + \|\partial_t u(t,\,\cdot\,)\|^2_{L^2(|x|<R_0)} \Big)\, dt \lesssim \|f\|^2_{H^1} + \|g\|^2_{L^2} + \int_0^\infty \|F(s,\,\cdot\,)\|^2_{L^2}\, ds, \tag{1.2}$$

whenever u is a solution of (1.1) *with data $(f(x), g(x))$ and forcing term $F(t,x)$ that both vanish for $|x| > R_0$.*

Here $A \lesssim B$ means that A is bounded by a constant times B, and, in what follows, the constant might change at each occurrence. Also, $\|h\|_{H^1(|x|<R_0)}$ denotes the L^2-norm of h and $\nabla_x h$ over the set $\{x \in \Omega :\, |x| < R_0\}$.

Let us review some important cases where the assumption (1.2) is valid. First of all, results from Vainberg [39], combined with the propagation of singularity results of Melrose and Sjöstrand [24], imply that if Δ is the standard Euclidean Laplacian and Ω is nontrapping, then if u is a solution of (1.1) with data of fixed compact support and forcing term $F \equiv 0$, then with $u' = (\partial_t u, \nabla_x u)$,

$$\|u'(t,\,\cdot\,)\|_{L^2(|x|<R_0)} \le \alpha(t) \|u'(0,\,\cdot\,)\|_{L^2},$$

where $\alpha(t) = O((1+t)^{-(n-1)})$ for either the Dirichlet-wave equation or the Neumann-wave equation when $n \ge 3$. For $n = 2$, if $\partial\Omega$ is assumed to be nonempty one has $\alpha(t) = O((\log(2+t))^{-2}(1+t)^{-1})$ for the Dirichlet-wave equation. Here we have used that, due to the Dirichlet boundary conditions and the fundamental theorem of calculus, the local L^2 norm can be controlled by the local L^2 norm of the gradient. Since these bounds yield $\alpha(t) \in L^1(\mathbb{R}_+)$, we conclude that Hypothesis B is valid in these cases. We remark that when $\Omega = \mathbb{R}^2$ and $\Delta_{\mathrm{g}} = \Delta$, then $\alpha \approx t^{-1}$ for large t (see [29]), and so, in this case, $\alpha \notin L^1(\mathbb{R}_+)$. Proofs of these results for $n \ge 3$ can be found in Melrose [23] and Ralston [29], while the result for the Dirichlet-wave equation for $n = 2$ follows from Vainberg [39] (see §4 and Remark 4 on p. 40). [1]

For the case where Δ_{g} is assumed to be a time-independent variable coefficient compact perturbation of Δ and Ω is assumed to be nontrapping with respect to the metric associated with Δ_{g}, one also has that (1.2) is valid for the Dirichlet-wave equation for all $n \ge 3$ as well for $n = 2$ if $\partial\Omega \ne \emptyset$. See Taylor [38] and Burq [4].

Having described the main assumption about the linear problem, let us now describe the nonlinear equations that we shall consider. They are of the form

$$\begin{cases} (\partial_t^2 - \Delta_{\mathrm{g}})u(t,x) = F_p\big(u(t,x)\big), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ Bu = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \quad x \in \Omega, \end{cases} \tag{1.3}$$

with B as above. We shall assume that the nonlinear term behaves like $|u|^p$ when u is small, and so we assume that

$$\sum_{0 \le j \le 2} |u|^j \, \big| \partial_u^j F_p(u) \big| \lesssim |u|^p \tag{1.4}$$

[1] We are very grateful to Jim Ralston for patiently explaining these results and their history to us.

when u is small.

We shall be assuming that the data (and some of its derivatives) are small in certain Sobolev norms that we now describe.

As in the earlier works that proved global Strichartz estimates ([4], [25], [33]), we shall restrict ourselves to the case where the Sobolev index γ is smaller than $n/2$. One reason for this is that the Strichartz estimates that seem to arise in applications always have $\gamma \le 1$. Another reason is that when $|\gamma| < n/2$, multiplication by a smooth function $\beta \in C_0^\infty(\mathbb{R}^n)$ is continuous from $\dot H^\gamma(\mathbb{R}^n)$ to $H^\gamma(\mathbb{R}^n)$ and the two norms are equivalent on functions with fixed compact support. Recall that $\dot H^\gamma(\mathbb{R}^n)$ is the homogeneous Sobolev space with norm given by

$$\|f\|^2_{\dot H^\gamma(\mathbb{R}^n)} = \big\| (\sqrt{-\Delta})^\gamma f \big\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \big| \, |\xi|^\gamma \hat f(\xi) \, \big|^2 \, d\xi,$$

while the inhomogeneous Sobolev space $H^\gamma(\mathbb{R}^n)$ has norm defined by

$$\|f\|^2_{H^\gamma(\mathbb{R}^n)} = \big\| \, (1-\Delta)^{\gamma/2} f \, \big\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \big| \, (1+|\xi|^2)^{\gamma/2} \hat f(\xi) \, \big|^2 \, d\xi,$$

with $\hat f$ denoting the Fourier transform and Δ denoting the standard Laplacian.

Let us now describe the Sobolev spaces on Ω that we shall consider. Let β be a smooth cutoff on $\mathbb{R}^n$ with β and $1-\beta$ respectively supported where $|x| < 2R$ and $|x| > R$. Let Ω' be the embedding of $\Omega \cap \{|x| < 2R\}$ into the torus obtained by periodic extension of $\Omega \cap [-2R, 2R]^n$, so that $\partial\Omega' = \partial\Omega$. We define

$$\begin{aligned} \|f\|_{H^\gamma_B(\Omega)} &= \|\beta f\|_{H^\gamma_B(\Omega')} + \|(1-\beta) f\|_{H^\gamma(\mathbb{R}^n)} \\ \|f\|_{\dot H^\gamma_B(\Omega)} &= \|\beta f\|_{H^\gamma_B(\Omega')} + \|(1-\beta) f\|_{\dot H^\gamma(\mathbb{R}^n)} , \qquad |\gamma| < n/2 \, . \end{aligned}$$

The spaces $H^\gamma_B(\Omega')$ are defined by a spectral decomposition of $\Delta_{\rm g}|_{\Omega'}$ subject to the boundary condition B. In the homogeneous spaces $\dot H^\gamma_B(\Omega)$ it is assumed that $(1-\beta)f$ belongs to $\dot H^\gamma(\mathbb{R}^n)$, so that the Sobolev embedding $\dot H^\gamma_B(\Omega) \hookrightarrow L^p(\Omega)$ holds with $p = 2n/(n-2\gamma)$. From this, it is verified that the Sobolev spaces on Ω are independent of the choice of β and R, and thus the $\dot H^\gamma_B(\Omega)$ and $H^\gamma_B(\Omega)$ norms are equivalent on functions of fixed bounded support. We note that $H^{-\gamma}_B(\Omega)$ is the dual of $H^\gamma_B(\Omega)$, and $\dot H^{-\gamma}_B(\Omega)$ is dual to $\dot H^\gamma_B(\Omega)$ for $|\gamma| < n/2$. Also, for γ a nonnegative integer,

$$\begin{aligned} \|f\|^2_{H^\gamma_B(\Omega)} &\approx \sum_{|\alpha| \le \gamma} \|\partial_x^\alpha f\|^2_{L^2(\Omega)} \\ \|f\|^2_{\dot H^\gamma_B(\Omega)} &\approx \sum_{|\alpha| = \gamma} \|\partial_x^\alpha f\|^2_{L^2(\Omega)} . \end{aligned}$$

The Sobolev spaces as defined are verified to be an analytic interpolation scale of spaces. The above definition then agrees, for nonnegative integer γ, with the subspace of $H^\gamma(\overline{\Omega})$ such that $B(\Delta_{\rm g}^j f) = 0$ for all j for which the trace is well defined, and for general γ by duality and interpolation. Finally, for every γ the set of functions $f \in C_0^\infty(\overline{\Omega})$ such that $B(\Delta_{\rm g}^j f) = 0$ for all $j \ge 0$ is dense in the norm.

Our hypotheses regarding the data in (1.3) will only involve certain $\gamma \in (0, \frac12)$, while the ones in the abstract Strichartz estimates to follow only involve certain $\gamma \le (n-1)/2$.

In practice the useful Strichartz-type estimates always involve $\gamma \in (0,1]$. This is the case for the mixed-norm Strichartz estimates of Keel and Tao [18] and others for the case $\Omega = \mathbb{R}^n$, $\Delta = \Delta_{\mathrm{g}}$, as well as for the mixed-norm estimates for (1.1) that we shall state.

The data (f,g) in Theorem 1.1 below will have second derivatives belonging to $\dot H^\gamma_B(\Omega)\times \dot H^{\gamma-1}_B(\Omega)$, where $\gamma\in(0,\frac12)$, thus will locally belong to $H^{2+\gamma}(\overline{\Omega})\times H^{1+\gamma}(\overline{\Omega})$. The boundary condition for (f,g) to locally belong to $H^{2+\gamma}_B(\Omega)\times H^{1+\gamma}_B(\Omega)$ for $\gamma\in(0,\frac12)$ is the same as for $H^2_B(\Omega)\times H^1_B(\Omega)$, which for the Dirichlet case is $f|_{\partial\Omega}=g|_{\partial\Omega}=0$, and for Neumann is $\partial_\nu f|_{\partial\Omega}=0$. These are the assumptions placed on the data (f,g) in Theorem 1.1.

If we let

$$\{Z\} = \{\partial_l,\, x_j\partial_k - x_k\partial_j : 1\le l\le n,\, 1\le j<k\le n\}$$

then we can now state our existence theorem for (1.3).

Theorem 1.1. *Let $n=3$ or 4, and fix $\Omega\subset\mathbb{R}^n$ and boundary operator B as above. Assume further that Hypothesis B is valid.*

Let $p=p_c$ be the positive root of

$$(n-1)p^2-(n+1)p-2=0, \tag{1.5}$$

and fix $p_c<p<(n+3)/(n-1)$. Then if

$$\gamma = \tfrac{n}{2}-\tfrac{2}{p-1}, \tag{1.6}$$

there is an $\varepsilon_0>0$ depending on Ω, B and p so that (1.3) has a global solution satisfying $(Z^\alpha u(t,\,\cdot\,),\partial_t Z^\alpha u(t,\,\cdot\,))\in \dot H^\gamma_B\times\dot H^{\gamma-1}_B$, $|\alpha|\le 2$, $t\in\mathbb{R}_+$, whenever the initial data satisfies the boundary conditions of order 2, and

$$\sum_{|\alpha|\le 2}\Big(\|Z^\alpha f\|_{\dot H^\gamma_B(\Omega)}+\|Z^\alpha g\|_{\dot H^{\gamma-1}_B(\Omega)}\Big)<\varepsilon \tag{1.7}$$

with $0<\varepsilon<\varepsilon_0$.

In the case where $\Omega=\mathbb{R}^n$ and $\Delta_{\mathrm{g}}=\Delta$ it is known that $p>p_c$ is necessary for global existence (see John [16], Glassey [12] and Sideris [31]). In this case under a somewhat more restrictive smallness condition global existence was established by John [16] for the case where $n=3$, then Glassey [12] for $n=2$, Zhou [40] for $n=4$, Lindblad and Sogge [22] for $n\le 8$ and then Georgiev, Lindblad and Sogge [11] for all n (see also Tataru [37]). For obstacles, when $n=4$, $\Delta_{\mathrm{g}}=\Delta$ the results in Theorem 1.1 for the Dirichlet-wave equation outside of nontrapping obstacles under a somewhat more restrictive smallness assumption was obtained in [8].

Also, when $\Omega=\mathbb{R}^3$, $\Delta_{\mathrm{g}}=\Delta$, it was shown in Sogge [35] that, for the spherically symmetric case, the variant of the condition (1.7) saying that the $\dot H^\gamma(\mathbb{R}^3)\times\dot H^{\gamma-1}(\mathbb{R}^3)$ norm of the data be small with γ as in (1.6) is sharp. Further work in this direction (for the non-obstacle case) was done by Hidano [13], [14] and Fang and Wang [10].

It is not difficult to see that the condition (1.7) is sharp in the sense that there are no global existence results for $\gamma>\frac n2-\frac{2}{p-1}$. To do this we use well known results concerning blowup solutions for $(\partial_t^2-\Delta)v=|v|^p$, $p>0$, in $\mathbb{R}_+\times\mathbb{R}^n$ (see Levine [19]). Specifically, we shall use the fact that given $\delta>0$ one can find C^∞_0 data (v_0,v_1) vanishing for $|x|<R$ so that the solution of $(\partial_t^2-\Delta)v=|v|^p$, $v(0,\,\cdot\,)=v_0, \partial_t v(0,\,\cdot\,)=v_1$ blows up within

time δ. Next, let us assume that the above global existence results for $(\Omega, B, \Delta_{\rm g})$ held for this nonlinearity and some $\gamma > \frac{n}{2} - \frac{2}{p-1}$ in (1.7). Then, if λ is sufficiently large, the $\dot H^\gamma_B \times \dot H^{\gamma-1}_B$ norm of $(\lambda^{-2/(p-1)}v_0(\,\cdot\,/\lambda),\, \lambda^{-1-2/(p-1)}v_1(\,\cdot\,/\lambda))$ would be bounded by its $\dot H^\gamma(\mathbb{R}^n)\times \dot H^{\gamma-1}(\mathbb{R}^n)$ norm, which equals $\lambda^{n/2-2/(p-1)-\gamma}\|(v_0,v_1)\|_{\dot H^\gamma(\mathbb{R}^n)\times \dot H^{\gamma-1}(\mathbb{R}^n)}$. Since this goes to zero as $\lambda\to\infty$ for $\gamma > \frac{n}{2} - \frac{2}{p-1}$, we conclude that if the above existence results held for this value of γ then we would obtain a global solution of $(\partial_t^2 - \Delta_{\rm g})u_\lambda = |u_\lambda|^p$, $u_\lambda(t,x) = 0, (t,x)\in \mathbb{R}_+\times\partial\Omega$ with initial data $(\lambda^{-2/(p-1)}v_0(\,\cdot\,/\lambda), \lambda^{-1-2/(p-1)}v_1(\,\cdot\,/\lambda))$. Since v_0 and v_1 vanish for $|x| < R$, by finite propagation speed, if $\delta > 0$ is small and fixed, then for large λ if we extend u_λ to be zero on Ω^c then the resulting function would agree with the solution of the Minkowski space wave equation $(\partial_t^2-\Delta)v_\lambda = |v_\lambda|^p$ on $[0,\delta\lambda]\times\mathbb{R}^n$ with data $(\lambda^{-2/(p-1)}v_0(\,\cdot\,/\lambda), \lambda^{-1-2/(p-1)}v_1(\,\cdot\,/\lambda))$. By scaling $v(t,x) = \lambda^{2/(p-1)}v_\lambda(\lambda t, \lambda x)$ would then solve the Minkowski space equation $(\partial_t - \Delta)v = |v|^p$ on $[0,\delta]\times\mathbb{R}^n$ with initial data $(v_0(x), v_1(x))$. As we noted before, we can always choose (v_0, v_1) so that this is impossible for a given $\delta > 0$, which allows us to conclude that the above existence results do not hold if the Sobolev exponent γ in (1.7) is larger than $\frac{n}{2} - \frac{2}{p-1}$.

As a final remark, we point out that we have restricted ourselves to the case where $p < (n+3)/(n-1)$ because of the techniques that we shall employ. However, since the solutions obtained are small, the above existence theorem leads to small-data global existence of (1.3) when p is larger than or equal to the conformal power $(n+3)/(n-1)$.

To prove Theorem 1.1, we shall use certain "abstract Strichartz estimates" which we now describe. Earlier works ([4], [25], [33]) have focused on establishing certain mixed norm, $L^q_tL^r_x$ estimates on $\mathbb{R}_+\times\Omega$ for solutions of (1.1). For certain applications, such as obtaining the Strauss conjecture in various settings, it is convenient to replace the L^r_x norm with a more general one. To this end, we consider pairs of normed function spaces $X(\mathbb{R}^n)$ and $X(\Omega)$. The spaces are localizable, in that $\|f\|_X \approx \|\beta f\|_X + \|(1-\beta)f\|_X$ for smooth, compactly supported β, with $\beta = 1$ on a neighborhood of $\mathbb{R}^n\backslash\Omega$ in case $X = X(\Omega)$. Finally, we assume that

$$\|(1-\beta)f\|_{X(\Omega)} \approx \|(1-\beta)f\|_{X(\mathbb{R}^n)} \tag{1.8}$$

for such β. Weighted mixed L^p spaces, as well as $\big(\dot H^\gamma(\mathbb{R}^n), \dot H^\gamma_B(\Omega)\big)$, are the examples used in the proof of Theorem 1.1.

We shall let $\|\cdot\|_{X'}$ denote the dual norm (respectively over $\mathbb{R}^n$ and Ω) so that

$$\|u\|_X = \sup_{\|v\|_{X'}=1}\Big|\int u\,\overline{v}\,dx\Big|.$$

An important example for us is when

$$\|u\|_X = \|\,|x|^\alpha u\|_{L^p},$$

for a given $1\le p\le\infty$ and $|\alpha| < n/p$, in which case the dual norm is

$$\|v\|_{X'} = \|\,|x|^{-\alpha}v\|_{L^{p'}},$$

with p' denoting the conjugate exponent.

We shall consider time Lebesgue exponents $q \ge 2$ and assume that we have the global Minkowski abstract Strichartz estimates

$$\|v\|_{L^q_t X(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|v(0,\,\cdot\,)\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\partial_t v(0,\,\cdot\,)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}\,, \tag{1.9}$$

assuming that

$$(\partial_t^2 - \Delta)v = 0 \quad \text{in } \ \mathbb{R}\times\mathbb{R}^n\,. \tag{1.10}$$

Here

$$\|v\|_{L^q_t X(I\times\mathbb{R}^n)} = \Big(\int_I \|v(t,\,\cdot\,)\|^q_X\, dt\Big)^{1/q}, \quad I\subset\mathbb{R}.$$

We shall also consider analogous norms on $I\times\Omega$, $I\subset\mathbb{R}$,

$$\|u\|_{L^q_t X(I\times\Omega)} = \Big(\int_I \|u(t,\,\cdot\,)\|^q_{X(\Omega)}\, dt\Big)^{1/q}.$$

In addition to Hypothesis B and (1.9), we shall assume that we have the local abstract Strichartz estimates for Ω:

$$\|u\|_{L^q_t X([0,1]\times\Omega)} \lesssim \|f\|_{\dot H^\gamma_B(\Omega)} + \|g\|_{\dot H^{\gamma-1}_B(\Omega)}, \tag{1.11}$$

assuming that u solves (1.1) with vanishing forcing term, i.e.,

$$(\partial_t^2 - \Delta_{\rm g})u = 0 \quad \text{in } \ [0,1]\times\Omega. \tag{1.12}$$

Definition 1.2. *When* (1.9) *and* (1.11) *hold we say that* (X,γ,q) *is an admissible triple.*

We can now state our main estimate.

Theorem 1.3. *Let* $n\ge 2$ *and assume that* (X,γ,q) *is an admissible triple with*

$$q>2 \quad \textit{and} \quad \gamma\in[-\tfrac{n-3}{2}, \tfrac{n-1}{2}]. \tag{1.13}$$

Then if Hypothesis B is valid and if u *solves* (1.1) *with* $(\partial_t^2-\Delta_{\rm g})u\equiv 0$*, we have the global abstract Strichartz estimates*

$$\|u\|_{L^q_t X(\mathbb{R}\times\Omega)} \lesssim \|f\|_{\dot H^\gamma_B(\Omega)} + \|g\|_{\dot H^{\gamma-1}_B(\Omega)}. \tag{1.14}$$

The condition on γ in (1.13) is the one to ensure that γ and $1-\gamma$ are both $\le (n-1)/2$, which is what the proof seems to require. Unfortunately, for $n=2$, this forces γ to be equal to $1/2$, while a larger range of $\gamma\in(0,1)$ is what certain applications require. For this reason, we are unable at present to show that the Strauss conjecture for obstacles holds when $n=2$. See the end of the next section for further discussion.

Corollary 1.4. *Assume that* (X,γ,q) *and* $(Y,1-\gamma,r)$ *are admissible triples and that Hypothesis B is valid. Also assume that* (1.14) *holds for* (X,γ,q) *and* $(Y,1-\gamma,r)$*, and that* $0\le\gamma\le 1$*. Then we have the following global abstract Strichartz estimates for the solution of* (1.1)

$$\|u\|_{L^q_t X(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\dot H^\gamma_B(\Omega)} + \|g\|_{\dot H^{\gamma-1}_B(\Omega)} + \|F\|_{L^{r'}_t Y'(\mathbb{R}_+\times\Omega)}, \tag{1.15}$$

where r' *denotes the conjugate exponent to* r *and* $\|\cdot\|_{Y'}$ *is the dual norm to* $\|\cdot\|_Y$.

For simplicity, in the corollary we have limited ourselves to the case where $0 \le \gamma \le 1$ since that is all that is needed for the applications.

Let us give the simple argument that shows that (1.15) follows from (1.14). To prove (1.15), we may assume by (1.14) that the initial data vanishes. By (1.14) and the Duhamel formula, if $P = \sqrt{-\Delta_{\rm g}}$ is the square root of minus the Laplacian (with the boundary conditions B), then we need show

$$\Big\| \int_0^t \sin\big((t-s)P\big)P^{-1}F(s,\,\cdot\,)\,ds\,\Big\|_{L_t^q X(\mathbb{R}_+\times\Omega)} \lesssim \|F\|_{L_t^{r'}Y'(\mathbb{R}_+\times\Omega)}\,.$$

Since $q > r'$, an application of the Christ-Kiselev lemma (cf. [7], [33], [36, chapter 4]) shows that it suffices to prove the estimate

$$\Big\| \int_0^\infty \sin\big((t-s)P\big)P^{-1}F(s,\,\cdot\,)\,ds\,\Big\|_{L_t^q X(\mathbb{R}_+\times\Omega)} \lesssim \|F\|_{L_t^{r'}Y'(\mathbb{R}_+\times\Omega)}\,.$$

After factorization of the sin function, it suffices by (1.14) to show that

$$\Big\| \int_0^\infty \cos(sP)F(s,\,\cdot\,)\,ds\,\Big\|_{\dot H^{\gamma-1}(\Omega)} + \Big\| \int_0^\infty P^{-1}\sin(sP)F(s,\,\cdot\,)\,ds\,\Big\|_{\dot H^{\gamma}(\Omega)} \lesssim \|F\|_{L_t^{r'}Y'(\mathbb{R}_+\times\Omega)}\,.$$

This, however, is the dual version of (1.14) for $(Y, 1-\gamma, r)$. □

As a special case of (1.15) when the spaces X and Y are the standard Lebesgue spaces, we have the following

Corollary 1.5. *Suppose that $n \ge 3$ and that Hypothesis B is valid. Suppose that $q, \tilde q > 2$, $r, \tilde r \ge 2$ and that*

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{\tilde q'} + \frac{n}{\tilde r'} - 2$$

and

$$\frac{2}{q} + \frac{n-1}{r}\,,\ \frac{2}{\tilde q} + \frac{n-1}{\tilde r} \le \frac{n-1}{2}.$$

Then if the local Strichartz estimate (1.11) *holds respectively for the triples $\big(L^r(\Omega), \gamma, q\big)$ and $\big(L^{\tilde r}(\Omega), 1-\gamma, \tilde q\big)$, it follows that when u solves* (1.1)

$$\|u\|_{L_t^q L_x^r(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\dot H_B^\gamma(\Omega)} + \|g\|_{\dot H_B^{\gamma-1}(\Omega)} + \|F\|_{L_t^{\tilde q'} L_x^{\tilde r'}(\mathbb{R}_+\times\Omega)}.$$

These results also hold for $n = 2$ under the above assumption, provided that $\gamma = 1/2$.

These estimates of course are the obstacle versions of the mixed-norm estimates for $\mathbb{R}^n$ and $\Delta_{\rm g} = \Delta$. When $n \ge 3$ (and (1.11) is valid) they include all the ones in the Keel-Tao theorem [18], excluding the cases where either q or $\tilde q$ is 2. For the Dirichlet-wave operator ($B = Id$) these results were proved in odd dimensions by Smith and Sogge [33] and then by Burq [4] and Metcalfe [25] for even dimensions. The Neumann case was not treated, but it follows from the same proof. Unfortunately, the known techniques seem to only apply to the case of $\gamma = 1/2$ when $n = 2$, and Hypothesis B seems also to require $B = Id$ and $\partial\Omega \ne \emptyset$ in this case. The restriction that $\gamma = 1/2$ when $n = 2$ comes from the second part of (1.13), while for $n \ge 3$ this is not an issue due to the fact that the Sobolev exponents γ in Corollary 1.5 always satisfy $0 \le \gamma \le 1$. Also, at present, the knowledge of the local Strichartz estimates (1.11) when $X = L^r(\Omega)$ is limited. When Ω is the exterior of a geodesically convex obstacle, they were obtained by Smith and Sogge [32]. Recently, there has been work on proving local Strichartz estimates when $X = L^r(\Omega)$ for more general exterior domains ([5], [6], [3], [34]), but only partial results for a more restrictive range of exponents than the ones described in Corollary 1.5 have been obtained.

2. **Proof of Abstract Strichartz Estimates.**

As mentioned before, we shall prove (1.14) by adapting the arguments from [4], [25] and [33]. We shall assume that (1.2) is valid for $(\Omega, \Delta_{\mathrm{g}})$ throughout. A key step in the proof of Theorem 1.3 will be to establish the following result that is implicit in [4].

Proposition 2.1. *Let w solve the inhomogeneous wave equation in Minkowski space*

$$\begin{cases} (\partial_t^2 - \Delta)w = F \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ w|_{t=0} = \partial_t w|_{t=0} = 0. \end{cases}$$

Assume as above that (1.9) is valid whenever v is a solution of the homogeneous wave equation (1.10). Assume further that $q > 2$ and $\gamma \geq -\frac{n-3}{2}$. Then, if

$$F(t,x) = 0 \quad \text{if} \quad |x| > 2R\,,$$

we have

$$\|w\|_{L_t^q X(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \|F\|_{L_t^2 H^{\gamma-1}(\mathbb{R}_+ \times \mathbb{R}^n)}.$$

At the end of this section we shall show that when $n = 2$ the assumption that $\gamma \geq 1/2$ when $n = 2$ is necessary even in the model case where $X = L^r(\mathbb{R}^n)$ with $2/q + 1/r = 1/2$ and $1/q + 2/r = 1 - \gamma$.

To prove Proposition 2.1, we shall use our free space hypothesis (1.9) and the following result from [33].

Lemma 2.2. *Fix $\beta \in C_0^\infty(\mathbb{R}^n)$ and assume that $\gamma \leq \frac{n-1}{2}$. Then*

$$\int_{-\infty}^{\infty} \Big\| \beta(\,\cdot\,)\big(e^{it|D|}f\big)(t,\,\cdot\,) \Big\|^2_{H^\gamma(\mathbb{R}^n)} \, dt \lesssim \|f\|^2_{\dot H^\gamma(\mathbb{R}^n)},$$

if $|D| = \sqrt{-\Delta}$.

As was shown in [33], this lemma just follows from an application of Plancherel's theorem and the Schwarz inequality. The assumption that $\gamma \leq (n-1)/2$ is easily seen to be sharp.

To prove Proposition 2.1, we note that since we are assuming that $q > 2$, by the Christ-Kiselev lemma [7], it suffices to show that

$$\Big\| \int_0^\infty e^{i(t-s)|D|} |D|^{-1} \beta(\,\cdot\,) G(s,\,\cdot\,)\, ds \Big\|_{L_t^q X(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \|G\|_{L_t^2 H^{\gamma-1}(\mathbb{R}^n)}, \tag{2.1}$$

assuming that $\beta \in C_0^\infty(\mathbb{R}^n)$. If we apply (1.9), we conclude that the left side of this inequality is majorized by

$$\Big\| \int_0^\infty e^{-is|D|} |D|^{-1+\gamma} \beta(\,\cdot\,) G(s,\,\cdot\,)\, ds \Big\|_{L^2(\mathbb{R}^n)}.$$

Since $\|(1-\Delta)^{(\gamma-1)/2}G(s,\,\cdot\,)\|_2 = \|G(s,\,\cdot\,)\|_{H^{\gamma-1}}$, it suffices to see that

$$\Big\| \int_0^\infty e^{-is|D|}|D|^{-1+\gamma}\beta(\,\cdot\,)(1-\Delta)^{(1-\gamma)/2}H(s,\,\cdot\,)\,ds\Big\|_{L^2(\mathbb{R}^n)} \lesssim \|H\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)}.$$

By duality, this is equivalent to the statement that

$$\big\|\,(1-\Delta)^{(1-\gamma)/2}\beta(\,\cdot\,)e^{is|D|}|D|^{-1+\gamma}h\big\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)} \lesssim \|h\|_{L^2(\mathbb{R}^n)}. \tag{2.2}$$

Since we are assuming that $\gamma \geq -\frac{n-3}{2}$, we have that $1-\gamma \leq \frac{n-1}{2}$. Therefore, (2.2) follows from Lemma 2.2, completing the proof of Proposition 2.1. □

To prove Theorem 1.3 we also need a similar result for solutions of the wave equation (1.1) for $(\Omega, B, \Delta_{\mathrm{g}})$.

Proposition 2.3. *Let u solve (1.1) and assume that*

$$f(x) = g(x) = F(t,x) = 0, \quad \text{when } |x| > 2R. \tag{2.3}$$

Then if (X,γ,q) is an admissible triple with $q>2$ and $\gamma \geq -\frac{n-3}{2}$ we have

$$\|u\|_{L^q_tX(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{H^\gamma_B} + \|g\|_{H^{\gamma-1}_B} + \|F\|_{L^2_tH^{\gamma-1}_B}. \tag{2.4}$$

The key ingredients in the proof are Proposition 2.1 and the following variant of (1.2), which holds for all $\gamma\in\mathbb{R}$, provided (2.3) holds, and $\beta \in C^\infty_c(\mathbb{R}^n)$ equals 1 on a neighborhood of $\mathbb{R}^n\backslash\Omega$:

$$\begin{aligned}\|\beta u\|_{L^\infty_tH^\gamma_B} + \|\beta\partial_t u\|_{L^\infty_tH^{\gamma-1}_B} + \|\beta u\|_{L^2_tH^\gamma_B} + \|\beta\partial_t u\|_{L^2_tH^{\gamma-1}_B}& \\ \lesssim \|f\|_{H^\gamma_B} + \|g\|_{H^{\gamma-1}_B} + \|F\|_{L^2_tH^{\gamma-1}_B}&.\end{aligned} \tag{2.5}$$

The L^2_t estimates in (2.5) on u follow from (1.2) and elliptic regularity arguments for $\gamma\in\mathbb{Z}$, and by interpolation for the remaining $\gamma\in\mathbb{R}$. The L^∞_t estimates then follow from energy estimates, duality, and elliptic regularity.

To prove (2.4), let us fix $\beta\in C^\infty_0(\mathbb{R}^n)$ satisfying $\beta(x)=1$, $|x|\leq 3R$ and write

$$u = v + w, \quad \text{where } v = \beta u,\ w = (1-\beta)u.$$

Then w solves the free wave equation

$$\begin{cases}(\partial_t^2-\Delta)w = [\beta,\Delta]u\\ w|_{t=0} = \partial_t w|_{t=0} = 0.\end{cases}$$

An application of Proposition 2.1 shows that $\|w\|_{L^q_tX}$ is dominated by $\|\rho u\|_{L^2_tH^\gamma_B}$ if ρ equals one on the support of β. Therefore, by (2.5), $\|w\|_{L^q_tX}$ is dominated by the right side of (2.4).

As a result, we are left with showing that if $v=\beta u$ then

$$\|v\|_{L^q_tX(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{H^\gamma_B} + \|g\|_{H^{\gamma-1}_B} + \|F\|_{L^2_tH^{\gamma-1}_B}, \tag{2.6}$$

assuming, as above, that (2.3) holds. To do this, fix $\varphi\in C^\infty_0((-1,1))$ satisfying $\sum_{j=-\infty}^\infty \varphi(t-j)=1$. For a given $j\in\mathbb{N}$, let $v_j=\varphi(t-j)v$. Then v_j solves

$$\begin{cases}(\partial_t^2-\Delta_{\mathrm{g}})v_j = -\varphi(t-j)[\Delta,\beta]u + [\partial_t^2,\varphi(t-j)]\beta u + \varphi(t-j)F\\ Bv_j(t,x) = 0, \quad x\in\partial\Omega\\ v_j(0,\,\cdot\,) = \partial_t v_j(0,\,\cdot\,) = 0,\end{cases}$$

while $v_0 = v - \sum_{j=1}^{\infty} v_j$ solves

$$\begin{cases} (\partial_t^2 - \Delta_{\rm g})v_0 = -\tilde\varphi[\Delta,\beta]u + [\partial_t^2,\tilde\varphi]\beta u + \tilde\varphi F \\ Bv_0(t,x) = 0, \quad x \in \partial\Omega \\ v_0|_{t=0} = f,\ \partial_t v_0|_{t=0} = g, \end{cases}$$

if $\tilde\varphi = 1 - \sum_{j=1}^{\infty} \varphi(t-j)$ if $t \geq 0$ and 0 otherwise. If we then let $G_j = (\partial_t^2 - \Delta_{\rm g})v_j$ be the forcing term for v_j, $j = 0, 1, 2, \dots$, then, by (2.5), we have that

$$\sum_{j=0}^{\infty} \|G_j\|^2_{L_t^2 H_B^{\gamma-1}(\mathbb{R}_+\times\Omega)} \lesssim \|f\|^2_{H_B^\gamma} + \|g\|^2_{H_B^{\gamma-1}} + \|F\|^2_{L_t^2 H_B^{\gamma-1}}.$$

By the local Strichartz estimates (1.11) and Duhamel, we get for $j = 1, 2, \dots$

$$\|v_j\|_{L_t^q X(\mathbb{R}_+\times\Omega)} \lesssim \int_0^\infty \|G_j(s,\,\cdot\,)\|_{H_B^{\gamma-1}}\, ds \lesssim \|G_j\|_{L_t^2 H_B^{\gamma-1}},$$

using Schwarz's inequality and the support properties of the G_j in the last step. Similarly,

$$\|v_0\|_{L_t^q X(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{H_B^\gamma} + \|g\|_{H_B^{\gamma-1}} + \|G_0\|_{L_t^2 H_B^{\gamma-1}}\,.$$

Since $q > 2$, we have

$$\|v\|^2_{L_t^q X(\mathbb{R}_+\times\Omega)} \lesssim \sum_{j=0}^{\infty} \|v_j\|^2_{L_t^q X(\mathbb{R}_+\times\Omega)}$$

and so we get

$$\|v\|^2_{L_t^q X} \lesssim \|f\|^2_{H_B^\gamma} + \|g\|^2_{H_B^{\gamma-1}} + \|F\|^2_{L_t^2 H^{\gamma-1}}$$

as desired, which finishes the proof of Proposition 2.3. □

End of Proof of Theorem 1.3: Recall that we are assuming that $(\partial_t^2 - \Delta_{\rm g})u = 0$. By Proposition 2.3 we may also assume that the initial data for u vanishes when $|x| < 3R/2$. We then fix $\beta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1$, $|x| \leq R$ and $\beta(x) = 0$, $|x| > 3R/2$ and write

$$u = u_0 - v = (1-\beta)u_0 + (\beta u_0 - v)\,,$$

where u_0 solves the Cauchy problem for the Minkowski space wave equation with initial data defined to be (f,g) if $x \in \Omega$ and 0 otherwise. By the free estimate (1.9) and (1.8), we can restrict our attention to $\tilde u = \beta u_0 - v$. But

$$(\partial_t^2 - \Delta_{\rm g})\tilde u = -[\Delta,\beta]u_0 \equiv G$$

is supported in $R < |x| < 2R$, and satisfies

$$\int_0^\infty \|G(t,\,\cdot\,)\|^2_{H_B^{\gamma-1}}\, dt \lesssim \|f\|^2_{\dot H_B^\gamma} + \|g\|^2_{\dot H_B^{\gamma-1}} \tag{2.7}$$

by Lemma 2.2 and the fact that G vanishes on a neighborhood of $\partial\Omega$. Note also that $\tilde u$ has vanishing initial data. Therefore, since Proposition 2.3 tells us that $\|\tilde u\|^2_{L_t^q X(\mathbb{R}_+\times\mathbb{R}^n)}$ is dominated by the left side of (2.7), the proof is complete. □

For future reference, we note that the preceeding steps establish the following generalization of (2.5), assuming that $\gamma \in [-\frac{n-3}{2}, \frac{n-1}{2}]$, and that $F(x) = 0$ for $|x| > R$:

$$\|u\|_{L_t^\infty \dot H_B^\gamma} + \|\partial_t u\|_{L_t^\infty \dot H_B^{\gamma-1}} + \|\beta u\|_{L_t^2 H_B^\gamma} + \|\beta\partial_t u\|_{L_t^2 H_B^{\gamma-1}} \lesssim \|f\|_{\dot H_B^\gamma} + \|g\|_{\dot H_B^{\gamma-1}} + \|F\|_{L_t^2 H_B^{\gamma-1}}. \tag{2.8}$$

In particular, f and g have no support restrictions. To see that (2.8) holds, first consider bounding the terms $\|\beta\partial_t^j u\|_{L_t^2 H_B^{\gamma-j}}$ for $j = 0,1$. For these terms, it suffices by (2.5) to consider $F = 0$ and $f, g = 0$ near $\partial\Omega$. Decomposing $u = (1-\beta)u_0 + \tilde u$ as above, we may use (2.5) and (2.7) to deduce the L_t^2 bounds in (2.8) for u. These bounds now yield

$$\|(\partial_t^2 - \Delta_{\mathrm g})(1-\beta)u\|_{L_t^2 H_B^{\gamma-1}} + \|(\partial_t^2 - \Delta_{\mathrm g})\beta u\|_{L_t^2 H_B^{\gamma-1}} \lesssim \|f\|_{\dot H_B^\gamma} + \|g\|_{\dot H_B^{\gamma-1}} + \|F\|_{L_t^2 H_B^{\gamma-1}}\,.$$

The L_t^∞ bounds on βu now follow from (2.5). Finally, $(1-\beta)u$ satisfies the Minkowski wave equation on $\mathbb{R}\times\mathbb{R}^n$, with initial data in $\dot H^\gamma \times \dot H^{\gamma-1}$, and driving force $\tilde F \in L_t^2 \dot H^{\gamma-1}$ which vanishes for $|x| \ge R$. The contribution to u from its initial data satisfies the L_t^∞ bounds as a result of homogeneous Sobolev bounds for the Minkowski wave group. The contribution from $\tilde F$ is bounded using Lemma 2.2 and duality.

Let us conclude this section by showing that when $n = 2$ the restriction in Proposition 2.1 that $\gamma \ge 1/2$ is necessary in the case where $X = L^r(\mathbb{R}^2)$. In this case, by the standard mixed-norm Strichartz estimates (see e.g. [18]), the hypotheses of the Proposition are satisfied when $0 \le \gamma < 3/4$, $1/q + 2/r = 1-\gamma$ and $2/q + 1/r \le 1/2$.

Since the hypotheses are satisfied, if the Proposition were valid for a given γ and $X = L^r(\mathbb{R}^2)$ as above, then the $L_t^q L_x^r(\mathbb{R}_+ \times \mathbb{R}^2)$ norm of

$$WF(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^2} e^{ix\cdot\xi} \frac{\sin(t-s)|\xi|}{|\xi|} \hat F(s,\xi)\, d\xi ds$$

would have to be bounded by the $L_t^2 H^{\gamma-1}$ norm of F if $F(t,x) = 0$ when $|x| > 1$. We shall take F to be a product $h_T(s)\beta(x)$ where $\beta \in C^\infty(\mathbb{R}^2)$ vanishes for $|x| > 1$ but satisfies $\hat\beta(0) = 1$, while h_T is an odd function supported in $[-T,T]$. For this choice of F we have

$$WF(t,x) = -i \int_{\mathbb{R}^2} e^{ix\cdot\xi} \cos(t|\xi|) \hat h_T(|\xi|)\hat\beta(\xi)\, d\xi/|\xi|, \quad \text{if } t > T.$$

Fix a nonzero function $\rho \in C^\infty(\mathbb{R})$ supported in $(1/2, 1)$. If we take h_T to be the odd function which equals $T^{-1/2}\rho(s/T)$ for positive s, then since h_T has a non-zero L^2 norm which is independent of T, if Proposition 2.1 were valid for an $L_t^q L_x^r$ space as above, then it would follow that

$$\begin{aligned} WF(t,x) &= -iT^{1/2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \cos(t|\xi|)\hat h_1(T|\xi|)\hat\beta(\xi)\, d\xi/|\xi| \\ &= -iT^{-1/2} \int_{\mathbb{R}^2} e^{i\frac{x}{T}\cdot\xi} \cos(\tfrac{t}{T}|\xi|)\, \hat h_1(|\xi|)\, \hat\beta(\xi/T)\, d\xi/|\xi| \end{aligned}$$

would belong to $L_t^q L_x^r([T,\infty)\times\mathbb{R}^2)$ with a bound independent of T. An easy calculation shows that this norm equals

$$T^{-1/2+1/q+2/r} \Big\| \int_{\mathbb{R}^2} e^{ix\cdot\xi} \cos(t|\xi|)\hat h_1(|\xi|)\hat\beta(\xi/T) \frac{d\xi}{|\xi|} \Big\|_{L_t^q L_x^r([1,\infty)\times\mathbb{R}^2)}.$$

Since our assumption that $\hat{\beta}(0) = 1$ implies that the last factor on the right tends to a positive constant, we conclude that if the conclusion of Proposition 2.1 were valid for $X = L^r(\mathbb{R}^2)$, then we would need that

$$\frac{1}{2} \geq \frac{1}{q} + \frac{2}{r} = 1 - \gamma.$$

This means that when $n = 2$, the assumption that $\gamma \geq 1/2$ in Proposition 2.1 is necessary.

3. The Strauss conjecture for nontrapping obstacles when $n = 3, 4$.

Let us start the proof of Theorem 1.1 by going over the Minkowski space results that will be used. These will form the assumption (1.9) of Theorem 1.3.

Lemma 3.1. *Let u solve the Minkowski wave equation*

$$(\partial_t^2 - \Delta)u = F, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n$$
$$u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g.$$

Then, for $2 \leq p \leq \infty$, and γ satisfying

$$\frac{1}{2} - \frac{1}{p} < \gamma < \frac{n}{2} - \frac{1}{p}, \qquad \textit{and} \qquad \frac{1}{2} < 1 - \gamma < \frac{n}{2}, \tag{3.1}$$

we have the following estimate

$$\begin{aligned} \Big\| |x|^{\frac{n}{2} - \frac{n+1}{p} - \gamma} u \Big\|_{L_t^p L_r^p L_\omega^2(\mathbb{R}_+ \times \mathbb{R}^n)} &\lesssim \|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \\ &\quad + \Big\| |x|^{-\frac{n}{2} + 1 - \gamma} F \Big\|_{L_t^1 L_r^1 L_\omega^2(\mathbb{R}_+ \times \mathbb{R}^n)}. \end{aligned} \tag{3.2}$$

Here, and in what follows, we are using the mixed-norm notation with respect to the volume element

$$\|h\|_{L_r^q L_\omega^p} = \Big(\int_0^\infty \Big(\int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \Big)^{q/p} r^{n-1} dr \Big)^{1/q}$$

for finite exponents and

$$\|h\|_{L_r^\infty L_\omega^p} = \sup_{r>0} \Big(\int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \Big)^{1/p}.$$

We first note that, by the trace lemma for the unit sphere and scaling, we have

$$\sup_{r>0} r^{\frac{n}{2} - s} \Big(\int_{S^{n-1}} |v(r\omega)|^2 \, d\sigma(\omega) \Big)^{1/2} \lesssim \|v\|_{\dot{H}^s(\mathbb{R}^n)}, \quad \frac{1}{2} < s < \frac{n}{2}, \tag{3.3}$$

where $d\sigma$ denotes the unit measure on S^{n-1}. Consequently,

$$\sup_{r>0} r^{\frac{n}{2} - s} \Big(\int_{S^{n-1}} \big| \big(e^{it|D|}\varphi\big)(r\omega) \big|^2 \, d\sigma(\omega) \Big)^{1/2} \lesssim \|\varphi\|_{\dot{H}^s(\mathbb{R}^n)}, \quad \frac{1}{2} < s < \frac{n}{2},$$

which is equivalent to

$$\| \, |x|^{-\alpha} e^{it|D|} \varphi \|_{L_r^\infty L_\omega^2} \lesssim \|\varphi\|_{\dot{H}^{\frac{n}{2} + \alpha}(\mathbb{R}^n)}, \quad -\frac{n-1}{2} < \alpha < 0. \tag{3.4}$$

Note that by applying (3.3) to the Fourier transform of v, we see that it is equivalent to the uniform bounds

$$\Big(\int_{S^{n-1}} |\hat{v}(\lambda\omega)|^2 \, d\sigma(\omega)\Big)^{1/2} \lesssim \lambda^{-\frac{n}{2}+s} \| \, |x|^s v\|_{L^2(\mathbb{R}^n)}\,, \quad \lambda>0\,, \quad \frac{1}{2}<s<\frac{n}{2}\,,$$

which by duality is equivalent to

$$\Big\| \, |x|^{-s} \int_{S^{n-1}} h(\omega) e^{i\lambda x\cdot\omega} \, d\sigma(\omega) \Big\|_{L^2_x(\mathbb{R}^n)} \lesssim \lambda^{s-\frac{n}{2}} \|h\|_{L^2_\omega(S^{n-1})}\,, \tag{3.5}$$

for $\lambda>0$ and fixed $1/2<s<n/2$. Using this estimate we can obtain

$$\| \, |x|^{-s} e^{it|D|}\varphi\|_{L^2(\mathbb{R}_+\times\mathbb{R}^n)} \lesssim \| \, |D|^{s-\frac{1}{2}}\varphi\|_{L^2(\mathbb{R}^n)}, \quad \frac{1}{2}<s<\frac{n}{2}\,, \tag{3.6}$$

for, by after Plancherel's theorem with respect to the t-variable, we find that the square of the left side of (3.6) equals

$$\begin{aligned} (2\pi)^{-1} \int_0^\infty \int_{\mathbb{R}^n} \Big| \, |x|^{-s} \int_{S^{n-1}} e^{ix\cdot\rho\omega} \rho^{n-1} \hat{\varphi}(\rho\omega) \, d\sigma(\omega) \Big|^2 \, dx \, d\rho \\ \lesssim \int_0^\infty \int_{S^{n-1}} \rho^{2(n-1)} |\hat{\varphi}(\rho\omega)|^2 \, \rho^{2s-n} \, d\sigma(\omega) d\rho = \| \, |D|^{s-\frac{1}{2}}\varphi\|^2_{L^2(\mathbb{R}^n)}, \end{aligned}$$

using (3.5) in the first step.

If we interpolate between (3.4) and (3.6) we conclude that, for $2\le q\le\infty$,

$$\Big\| \, |x|^{\frac{n}{2}-\frac{n+1}{q}-\gamma} e^{it|D|}\varphi \Big\|_{L^q_t L^q_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^n)} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)}, \quad \frac{1}{2}-\frac{1}{q}<\gamma<\frac{n}{2}-\frac{1}{q}. \tag{3.7}$$

This estimate in turn implies that if v solves the Cauchy problem $(\partial_t^2-\Delta)v=0$ in $\mathbb{R}_+\times\mathbb{R}^n$ then

$$\begin{aligned} \Big\| \, |x|^{\frac{n}{2}-\frac{n+1}{q}-\gamma} v \Big\|_{L^q_t L^q_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^n)} \\ \lesssim \|v(0,\,\cdot\,)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t v(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}, \quad \frac{1}{2}-\frac{1}{q}<\gamma<\frac{n}{2}-\frac{1}{q}\,. \end{aligned} \tag{3.8}$$

The estimate dual to (3.3) is

$$\|\varphi\|_{\dot{H}^{\gamma-1}} \le \Big\| \, |x|^{-\frac{n}{2}+1-\gamma}\varphi\Big\|_{L^1_r L^2_\omega}\,. \tag{3.9}$$

By the Duhamel formula and (3.8)-(3.9), we then have

$$\begin{aligned} \Big\| \, |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \Big\|_{L^p_t L^p_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^n)} \lesssim \|u(0,\,\cdot\,)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} \\ + \Big\| \, |x|^{-\frac{n}{2}+1-\gamma}(\partial_t^2-\Delta)u \Big\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^n)}, \end{aligned} \tag{3.10}$$

provided that γ and $1-\gamma$ satisfy the condition in (3.8) for q equal to p and ∞, respectively, i.e., (3.1). □

A calculation shows that if

$$\gamma = \frac{n}{2}-\frac{2}{p-1}, \tag{3.11}$$

and

$$p_c<p<(n+3)/(n-1)$$

then (3.1) holds: $p > p_c$ is needed for the first part, and $p < (n+3)/(n-1)$ for the second. Additionally, as far as the powers of $|x|$ go in (3.10), we have

$$p\Big(\frac{n}{2}-\frac{n+1}{p}-\gamma\Big)=p\Big(\frac{(n+1)-(n-1)p}{p(p-1)}\Big)=-\frac{n}{2}+1-\gamma\,,\quad \text{if } \gamma=\frac{n}{2}-\frac{2}{p-1}. \tag{3.12}$$

As a result, by the arguments to follow, (3.2) is strong enough to show that for the non-obstacle, Minkowski space case, i.e. $\Omega=\mathbb{R}^n$, $\Delta_{\rm g}=\Delta$, if $2\le n\le 4$ then for $p_c<p<(n+3)/(n-1)$, the equation (1.3) has a global solution for small data as described in Theorem 1.3.

To prove the obstacle version of this result for $n=3$ and 4 we shall use a slightly weaker inequality for which it will be easy to show that we have the corresponding local Strichartz estimates (1.11) for $(\Omega,\Delta_{\rm g})$. To this end, if R is chosen so that $\partial\Omega$ is contained in $|x|<R$ and $\Delta=\Delta_{\rm g}$ for $|x|\ge R$ then we define $X=X_{\gamma,q}(\mathbb{R}^n)$ to be the space with norm defined by

$$\|h\|_{X_{\gamma,q}}=\|h\|_{L^{s_\gamma}(|x|<2R)}+\big\|\,|x|^{\frac n2-\frac{n+1}{q}-\gamma}h\|_{L^q_rL^2_\omega(|x|>2R)},\quad \text{if } n\big(\tfrac12-\tfrac1{s_\gamma}\big)=\gamma. \tag{3.13}$$

We then prove the following obstacle variant of (3.2).

Lemma 3.2. *For solutions of* (1.1) *if* $n\ge 3$ *and* $p>2$:

$$\begin{aligned}\Big\|\,|x|^{\frac n2-\frac{n+1}{p}-\gamma}u\Big\|_{L^p_tL^p_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}&+\|u\|_{L^p_tL^{s_\gamma}_x(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})}\\ \lesssim \|f\|_{\dot H^\gamma_B}+\|g\|_{\dot H^{\gamma-1}_B}&+\Big\|\,|x|^{-\frac n2+1-\gamma}F\Big\|_{L^1_tL^1_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}\\ &+\|F\|_{L^1_tL^{s'_{1-\gamma}}_x(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\end{aligned} \tag{3.14}$$

provided that (3.1) *holds.*

By (3.8) and Lemma 2.2 we have that the assumption (1.9) of Theorem 1.3 is valid if $1/2-1/q<\gamma<n/2-1/q$ and $2\le q\le\infty$, i.e.

$$\begin{aligned}\|v\|_{L^q_tX_{\gamma,q}(\mathbb{R}_+\times\mathbb{R}^n)}\lesssim \|v(0,\,\cdot\,)\|_{\dot H^\gamma(\mathbb{R}^n)}+\|\partial_tv(0,\,\cdot\,)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)},&\\ \text{if }\ (\partial_t^2-\Delta)v=0\ \text{ in }\ \mathbb{R}_+\times\mathbb{R}^n,&\end{aligned} \tag{3.15}$$

under the additional assumption that $\gamma\le(n-1)/2$ (which is the case for (3.11)). Indeed the contribution of the second part of the norm in (3.13) is controlled by (3.8). To handle the contribution of the first term in the right side of (3.13) we note that if $\beta\in C^\infty_0(\mathbb{R}^n)$ equals one when $|x|\le 3R$ then Sobolev estimates yield

$$\|v(t,\,\cdot\,)\|_{L^{s_\gamma}(|x|<2R)}\lesssim\|\beta(\,\cdot\,)v(t,\,\cdot\,)\|_{\dot H^\gamma(\mathbb{R}^n)}.$$

Thus, $\|v\|_{L^q_tL^{s_\gamma}(\mathbb{R}_+\times\{|x|<R\})}$ is controlled by the right side of (3.15) for $q=2$, by Lemma 2.2. Since this is also the case for $q=\infty$ by energy estimates, by interpolation we conclude that we can control the contribution of the first term in the right side of (3.13) to (3.15), which finishes the proof of (3.15).

Since the dual norm of $\|\,|x|^\alpha h\|_p$ is $\|\,|x|^{-\alpha}h\|_{p'}$, by Corollary 1.4, we would get (3.14) from (3.15) and Hypothesis B if we could show that for $q>2$

$$\|u\|_{L^q_tX_{\gamma,q}([0,1]\times\Omega)}\lesssim\|f\|_{\dot H^\gamma_B}+\|g\|_{\dot H^{\gamma-1}_B},$$

whenever u solves (1.1) with $F \equiv 0$, and, as above, $1/2 - 1/q < \gamma < n/2 - 1/q$. By the finite propagation speed of the wave equation, it is clear that the contribution of the second term in the right side of (3.13) will enjoy this estimate. As before, the first term satisfies it because of Sobolev estimates. This completes the proof of (3.14). □

Let us also observe a related estimate

$$\begin{aligned} \|u\|_{L_t^\infty \dot H_B^\gamma(\mathbb{R}_+\times\Omega)} + \|\partial_t u\|_{L_t^\infty \dot H_B^{\gamma-1}(\mathbb{R}_+\times\Omega)} + \|u\|_{L_t^\infty L_x^{s_\gamma}(\mathbb{R}_+\times\Omega)} + \|\beta u\|_{L_t^2 H_B^\gamma(\mathbb{R}_+\times\Omega)} \\ \lesssim \|f\|_{\dot H_B^\gamma} + \|g\|_{\dot H_B^{\gamma-1}} + \Big\| |x|^{-\frac{n}{2}+1-\gamma} F\Big\|_{L_t^1 L_r^1 L_\omega^2(\mathbb{R}_+\times\{|x|>2R\})} \\ + \|F\|_{L_t^1 L_x^{s'_{1-\gamma}}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}, \end{aligned} \tag{3.16}$$

assuming that (3.1) holds. Indeed, this is a direct consequence of (2.8) and the Duhamel formula, together with the inclusion $\dot H_B^\gamma(\Omega) \hookrightarrow L^{s_\gamma}(\Omega)$, and the following consequence of (3.9), and the dual estimate to Sobolev embedding $\dot H_B^{1-\gamma}(\Omega) \hookrightarrow L^{s_{1-\gamma}}(\Omega)$,

$$\|g\|_{\dot H_B^{\gamma-1}} \lesssim \| |x|^{-\frac{n}{2}+1-\gamma} g\|_{L_r^1 L_\omega^2(|x|>2R)} + \|g\|_{L^{s'_{1-\gamma}}(x\in\Omega:|x|<2R)}. \tag{3.17}$$

To prove Theorem 1.1 we shall require a variation of the last two estimates involving the vector fields

$$\{\Gamma\} = \{\partial_t, Z\}$$

where, as before, $\{Z\}$ are the vector fields $\{\partial_i,\, x_j\partial_k - x_k\partial_j : 1 \le i \le n, 1 \le j < k \le n\}$. Note that all the $\{\Gamma\}$ commute with $\Box_g = \partial_t^2 - \Delta_{\mathrm{g}}$ when $|x| > R$ because $\partial\Omega \subset \{x : |x| < R\}$ and $\Delta = \Delta_{\mathrm{g}}$ for $|x| > R$.

The main estimate we require is the following.

Lemma 3.3. *With p and γ as in Lemma 3.2, u solving (1.1) with $n \ge 3$, and (f, g, F) satisfying $H_B^2 \times H_B^1 \times H_B^1$ boundary conditions, then*

$$\begin{aligned} &\sum_{|\alpha|\le 2} \Big(\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} \Gamma^\alpha u\|_{L_t^p L_r^p L_\omega^2(\mathbb{R}_+\times\{|x|>2R\})} + \|\Gamma^\alpha u\|_{L_t^p L_x^{s_\gamma}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})} \Big) \\ &\lesssim \sum_{|\alpha|\le 2} \Big(\|Z^\alpha f\|_{\dot H_B^\gamma} + \|Z^\alpha g\|_{\dot H_B^{\gamma-1}} \Big) \\ &+ \sum_{|\alpha|\le 2} \Big(\| |x|^{-\frac{n}{2}+1-\gamma} \Gamma^\alpha F\|_{L_t^1 L_r^1 L_\omega^2(\mathbb{R}_+\times\{|x|>2R\})} + \|\Gamma^\alpha F\|_{L_t^1 L_x^{s'_{1-\gamma}}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})} \Big). \end{aligned} \tag{3.18}$$

The boundary conditions on (f, g, F) imply that $\partial_t^j u$ is locally in $H^{2+\gamma-j}(\Omega)$, $j = 0, 1, 2$, which will be implicitly used in elliptic regularity arguments. We will also use the fact that the Cauchy data for $\Gamma^\alpha u$ is bounded in $\dot H_B^\gamma \times \dot H_B^{\gamma-1}$ by the right hand side of (3.18) for $|\alpha| \le 2$. This is clear if Γ^α is replaced by Z^α. On the other hand, the Cauchy data for $\partial_t u$ is $(g, \Delta_{\mathrm{g}} f + F(0,\,\cdot))$. We may control

$$\sum_{|\alpha|\le 1} \Big(\|Z^\alpha g\|_{\dot H_B^\gamma} + \|Z^\alpha \Delta_{\mathrm{g}} f\|_{\dot H_B^{\gamma-1}} \Big) \le \sum_{|\alpha|\le 2} \Big(\|Z^\alpha f\|_{\dot H_B^\gamma} + \|Z^\alpha g\|_{\dot H_B^{\gamma-1}} \Big).$$

Recall that $\gamma \in (0, \frac{1}{2})$, so that $\dot{H}^\gamma_B(\Omega) = \dot{H}^\gamma(\overline{\Omega})$. To control the term $F(0,\,\cdot\,)$, we recall that $\Gamma = \{\partial_t, Z\}$, and use the bound

$$\sum_{|\alpha|\le 1} \|\Gamma^\alpha F\|_{L^\infty_t \dot{H}^{\gamma-1}_B(\mathbb{R}_+\times\Omega)} \le \sum_{|\alpha|\le 2} \|\Gamma^\alpha F\|_{L^1_t \dot{H}^{\gamma-1}_B(\mathbb{R}_+\times\Omega)} \tag{3.19}$$

which by (3.17) is seen to be dominated by the right hand side of (3.18). Similar considerations apply to the Cauchy data for $\partial_t^2 u$.

Let us now give the argument for (3.18). We first fix $\beta_0 \in C^\infty_0$ satisfying $\beta_0 = 1$ for $|x| \le R$ and supp$\beta_0 \subset \{|x| < 2R\}$. Then the first step in the proof of (3.18) will be to show that

$$\begin{aligned} &\sum_{|\alpha|\le 2}\Big(\|\,|x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma}(1-\beta_0)\Gamma^\alpha u\|_{L^p_t L^p_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} + \|(1-\beta_0)\Gamma^\alpha u\|_{L^p_t L^{s_\gamma}_x(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big) \\ &\lesssim \sum_{|\alpha|\le 2}\Big(\|Z^\alpha f\|_{\dot{H}^\gamma_B} + \|Z^\alpha g\|_{\dot{H}^{\gamma-1}_B}\Big) \\ &\quad + \sum_{|\alpha|\le 2}\Big(\|\,|x|^{-\frac{n}{2}+1-\gamma}\Gamma^\alpha F\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} + \|\Gamma^\alpha F\|_{L^1_t L^{s'_{1-\gamma}}_x(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big). \end{aligned} \tag{3.20}$$

Since the Γ commute with $\Box_g$ when $|x| \ge R$, we have

$$\Box_g\big((1-\beta_0)\Gamma^\alpha u\big) = (1-\beta_0)\Gamma^\alpha F - [\beta_0, \Delta_{\rm g}]\Gamma^\alpha u\,.$$

We can therefore write $(1-\beta_0)\Gamma^\alpha u$ as $v + w$ where $\Box_g v = (1-\beta_0)\Gamma^\alpha F$ and v has initial data $\big((1-\beta_0)\Gamma^\alpha u(0,\,\cdot\,), \partial_t(1-\beta_0)\Gamma^\alpha u(0,\,\cdot\,)\big)$, while $\Box_g w = -[\beta_0, \Delta_{\rm g}]\Gamma^\alpha u$ and w has vanishing initial data. If we do this, it follows by (3.14) that if for $|\alpha| \le 2$ we replace the term involving $(1-\beta_0)\Gamma^\alpha u$ by v in the left side of (3.20), then the resulting expression is dominated by the right side of (3.20). If we use (2.4), we find that if we replace $(1-\beta_0)\Gamma^\alpha u$ by w then the resulting expression is dominated by

$$\sum_{|\alpha|\le 2} \|\,[\beta_0, \Delta_{\rm g}]\Gamma^\alpha u\|_{L^2_t H^{\gamma-1}_B} \lesssim \sum_{j\le 2} \|\beta_1 \partial_t^j u\|_{L^2_t H^{\gamma+2-j}_B} \tag{3.21}$$

assuming that β_1 equals one on supp(β_0) and is supported in $|x| < 2R$. As a result, we would be done with the proof of (3.20) if we could show that the right hand side of (3.21) is dominated by the right side of (3.20). By (3.16) we control $\|\beta_1\partial_t^2 u\|_{L^2_t H^\gamma_B}$ by the right hand side of (3.20). On the other hand,

$$\|\beta_1 \partial_t u\|^2_{L^2_t H^{\gamma+1}_B} \lesssim \|\beta_1 \partial_t^2 u\|_{L^2_t H^\gamma_B}\, \|\beta_1 u\|_{L^2_t H^{\gamma+2}_B}$$

so it suffices to dominate $\|\beta_1 u\|_{L^2_t H^{\gamma+2}_B}$. Since $\Delta_{\rm g} u = \partial_t^2 u - F$, then if β_2 equals one on supp(β_1) and is supported in the set where $|x| < 2R$, we may use elliptic regularity and the equation to bound

$$\begin{aligned} \|\beta_1 u\|_{L^2_t H^{\gamma+2}_B} &\lesssim \|\beta_2 \Delta_{\rm g} u\|_{L^2_t H^\gamma_B} + \|\beta_2 u\|_{L^2_t H^\gamma_B} \\ &\lesssim \|\beta_2 \partial_t^2 u\|_{L^2_t H^\gamma_B} + \|\beta_2 u\|_{L^2_t H^\gamma_B} + \|\beta_2 F\|_{L^2_t H^\gamma_B}. \end{aligned}$$

The first two terms are dominated as above using (3.16). On the other hand, Sobolev embedding and duality yields

$$\begin{aligned}\|\beta_2 F\|_{L^2_t H^\gamma_B} &\lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha_x F\|_{L^2_t L^{s'_{1-\gamma}}(\mathbb{R}_+\times\{x\in\Omega:|x|\le 2R\})} \\ &\lesssim \sum_{|\alpha|\le 2} \|\partial^\alpha_{t,x} F\|_{L^1_t L^{s'_{1-\gamma}}(\mathbb{R}_+\times\{x\in\Omega:|x|\le 2R\})}.\end{aligned} \tag{3.22}$$

To finish the proof of (3.18), we need to show that the analog of (3.20) is valid when $(1-\beta_0)$ is replaced by β_0. Since the coefficients of Γ are bounded on $\mathrm{supp}(\beta_0)$, if β_1 equals one on $\mathrm{supp}(\beta_0)$ and is supported in $|x|<2R$, then by Sobolev embedding

$$\begin{aligned}\sum_{|\alpha|\le 2} \|\beta_0\Gamma^\alpha u\|_{L^p_t L^{s_\gamma}_x(\mathbb{R}_+\times\Omega)} &\lesssim \sum_{j\le 2} \|\beta_1\partial^j_t u\|_{L^p_t H^{\gamma+2-j}_B} \\ &\lesssim \sum_{j\le 2}\Bigl(\|\beta_1\partial^j_t u\|_{L^2_t H^{\gamma+2-j}_B} + \|\beta_1\partial^j_t u\|_{L^\infty_t H^{\gamma+2-j}_B}\Bigr).\end{aligned}$$

The terms in $L^2_t H^{\gamma+2-j}$ are dominated as above. To control the $L^\infty_t H^{\gamma+2-j}$ terms, and conclude the proof of (3.18), we establish the following estimate:

$$\begin{aligned}\sum_{|\alpha|\le 2} \|\Gamma^\alpha u\|_{L^\infty_t \dot H^\gamma_B} + \|\partial_t\Gamma^\alpha u\|_{L^\infty_t \dot H^{\gamma-1}_B} &\lesssim \sum_{|\alpha|\le 2}\Bigl(\|Z^\alpha f\|_{\dot H^\gamma_B(\Omega)} + \|Z^\alpha g\|_{\dot H^{\gamma-1}_B(\Omega)}\Bigr) \\ + \sum_{|\alpha|\le 2}\Bigl(\|\,|x|^{-\frac{n}{2}+(1-\gamma)}\Gamma^\alpha F\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} &+ \|\Gamma^\alpha F\|_{L^1_t L^{s'_{1-\gamma}}_x(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Bigr).\end{aligned} \tag{3.23}$$

The inequality where $\Gamma^\alpha u$ is replaced by $(1-\beta_0)\Gamma^\alpha u$ in (3.23) follows by energy estimates on $\mathbb{R}^n$, since the right hand side dominates $\|(1-\beta_0)\Gamma^\alpha F\|_{L^1_t \dot H^{\gamma-1}}$, together with (2.8) using the bound (3.21) to handle the commutator term. If $\Gamma^\alpha u$ is replaced on the left hand side by $\beta_0\Gamma^\alpha u$, the result is dominated by $\sum_{j\le 3}\|\beta_1\partial^j_t u\|_{L^\infty_t H^{2+\gamma-j}_B}$. For the case $j=0,1$, we write $\Box_{\mathrm{g}}(\beta_1 u) = \beta_1 F - [\Delta_{\mathrm{g}},\beta_1]u$, and use (2.5) with the Duhamel formula to bound

$$\begin{aligned}&\|\beta_1 u\|_{L^\infty_t H^{\gamma+2}_B} + \|\beta_1\partial_t u\|_{L^\infty_t H^{\gamma+1}_B} \\ &\qquad\lesssim \|\beta_1 f\|_{H^{\gamma+2}_B} + \|\beta_1 g\|_{H^{\gamma+1}_B} + \|\beta_2 u\|_{L^2_t H^{\gamma+2}_B} + \|\beta_1 F\|_{L^1_t H^{\gamma+1}_B}.\end{aligned}$$

The term on the right involving u is controlled previously; on the other hand, since F satisfies the $H^{\gamma+1}_B$ boundary conditions, then

$$\|\beta_1 F\|_{L^1_t H^{\gamma+1}_B} \lesssim \sum_{|\alpha|\le 2}\|\partial^\alpha_x F\|_{L^1_t L^{s'_{1-\gamma}}_x}.$$

To handle the terms for $j=2,3$ we use the equation to bound

$$\sum_{j=2,3}\|\beta_1\partial^j_t u\|_{L^\infty_t H^{2+\gamma-j}_B} \le \sum_{j=0,1}\Bigl(\|\beta_1\partial^j_t\Delta_{\mathrm{g}} u\|_{L^\infty_t H^{\gamma-j}_B} + \|\beta_1\partial^j_t F\|_{L^\infty_t H^{\gamma-j}_B}\Bigr).$$

The terms involving $\Delta_{\mathrm{g}} u$ are dominated by $\|\beta_2\partial^j_t u\|_{L^\infty_t H^{\gamma+2-j}_B}$ with $j=0,1$. The terms involving F are controlled for $j=1$ by (3.19), and for $j=0$ by observing that (3.22) holds with L^2_t replaced by L^∞_t. This completes the proof of (3.18) and (3.23). $\square$

We shall now use these estimates to prove Theorem 1.1.

Proof of Theorem 1.1: We assume Cauchy data (f,g) satsifying the smallness condition (1.7), and let u_0 solve the Cauchy problem (1.1) with $F=0$. We iteratively define u_k, for $k\ge 1$, by solving

$$\begin{cases}(\partial_t^2-\Delta_{\rm g})u_k(t,x)=F_p(u_{k-1}(t,x))\,, & (t,x)\in\mathbb{R}_+\times\Omega\\ u(0,\,\cdot\,)=f\\ \partial_t u(0,\,\cdot\,)=g\\ (Bu)(t,x)=0, \quad \text{on } \mathbb{R}_+\times\partial\Omega.\end{cases}$$

Our aim is to show that if the constant $\varepsilon>0$ in (1.7) is small enough, then so is

$$M_k=\sum_{|\alpha|\le 2}\Big(\big\|\Gamma^\alpha u_k\big\|_{L^\infty_t\dot H^\gamma_B(\mathbb{R}_+\times\Omega)}+\big\|\partial_t\Gamma^\alpha u_k\big\|_{L^\infty_t\dot H^{\gamma-1}_B(\mathbb{R}_+\times\Omega)}$$
$$+\big\|\,|x|^{\frac n2-\frac{n+1}{p}-\gamma}\Gamma^\alpha u_k\big\|_{L^p_tL^p_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}+\|\Gamma^\alpha u_k\|_{L^p_tL^{s_\gamma}_x(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})}\Big)$$

for every $k=0,1,2,\dots$

For $k=0$, it follows by (3.18) and (3.23) that $M_0\le C_0\varepsilon$, with C_0 a fixed constant. More generally, (3.18) and (3.23) yield that

$$M_k\le C_0\varepsilon+C_0\sum_{|\alpha|\le 2}\Big(\big\|\,|x|^{-\frac n2+1-\gamma}\Gamma^\alpha F_p(u_{k-1})\big\|_{L^1_tL^1_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}$$
$$+\|\Gamma^\alpha F_p(u_{k-1})\|_{L^1_tL_x^{s'_{1-\gamma}}(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})}\Big)\,. \tag{3.24}$$

Note that our assumption (1.4) on the nonlinear term F_p implies that for small v

$$\sum_{|\alpha|\le 2}|\Gamma^\alpha F_p(v)|\lesssim |v|^{p-1}\sum_{|\alpha|\le 2}|\Gamma^\alpha v|+|v|^{p-2}\sum_{|\alpha|\le 1}|\Gamma^\alpha v|^2\,.$$

Furthermore, since u_k will be locally of regularity $H^{\gamma+2}_B\subset L^\infty$ and F_p vanishes at 0, it follows that $F_p(u_k)$ satisfies the B boundary conditions if u_k does.

Since the collection Γ contains vectors spanning the tangent space to S^{n-1}, by Sobolev embedding for $n=3,4$ we have

$$\|v(r\,\cdot\,)\|_{L^\infty_\omega}+\sum_{|\alpha|\le 1}\|\Gamma^\alpha v(r\,\cdot\,)\|_{L^4_\omega}\lesssim\sum_{|\alpha|\le 2}\|\Gamma^\alpha v(r\,\cdot\,)\|_{L^2_\omega}\,.$$

Consequently, for fixed $t,r>0$

$$\sum_{|\alpha|\le 2}\|\Gamma^\alpha F_p(u_{k-1}(t,r\,\cdot\,))\|_{L^2_\omega}\lesssim\sum_{|\alpha|\le 2}\|\Gamma^\alpha u_{k-1}(t,r\,\cdot\,)\|^p_{L^2_\omega}\,.$$

By (3.12), the first summand in the right side of (3.24) is dominated by $C_1M^p_{k-1}$.

We next observe that, since $s_\gamma>2$ and $n\le 4$, it follows by Sobolev embedding on $\{\Omega\cap|x|<2R\}$ that

$$\|v\|_{L^\infty(x\in\Omega:|x|<2R)}+\sum_{|\alpha|\le 1}\|\Gamma^\alpha v\|_{L^4(x\in\Omega:|x|<2R)}\lesssim\sum_{|\alpha|\le 2}\|\Gamma^\alpha v\|_{L^{s_\gamma}(x\in\Omega:|x|<2R)}\,.$$

Since $s'_{1-\gamma} < 2$, it holds for each fixed t that

$$\sum_{|\alpha|\le 2} \|\Gamma^\alpha F_p(u_{k-1}(t,\,\cdot\,))\|_{L^{s'_{1-\gamma}}(x\in\Omega:|x|<2R)} \lesssim \sum_{|\alpha|\le 2} \|\Gamma^\alpha u_{k-1}(t,\,\cdot\,)\|^p_{L^{s_\gamma}(x\in\Omega:|x|<2R)}\,.$$

The second summand in the right side of (3.24) is thus also dominated by $C_1 M^p_{k-1}$, and we conclude that $M_k \le C_0\varepsilon + 2C_0\, C_1 M^p_{k-1}$. For ε sufficiently small, then

$$M_k \le 2\, C_0\varepsilon, \quad k = 1, 2, 3, \dots \tag{3.25}$$

To finish the proof of Theorem 1.1 we need to show that u_k converges to a solution of the equation (1.3). For this it suffices to show that

$$\begin{aligned} A_k = \big\|\, |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma}(u_k - u_{k-1})\,\big\|_{L^p_t L^p_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})} \\ + \|u_k - u_{k-1}\|_{L^p_t L^{s_\gamma}_x(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})} \end{aligned}$$

tends geometrically to zero as $k\to\infty$. Since $|F_p(v) - F_p(w)| \lesssim |v-w|(\,|v|^{p-1} + |w|^{p-1}\,)$ when v and w are small, the proof of (3.25) can be adapted to show that, for small $\varepsilon > 0$, there is a uniform constant C so that

$$A_k \le C A_{k-1}(M_{k-1} + M_{k-2})^{p-1},$$

which, by (3.25), implies that $A_k \le \frac{1}{2}A_{k-1}$ for small ε. Since A_1 is finite, the claim follows, which finishes the proof of Theorem 1.1. □

References

[1] M. Ben-Artzi: *Regularity and smoothing for some equations of evolution*, in "Nonliner Partial Differential Equations and Applications" (H. Brezis and J. L. Lions, eds.), London, 1994, pp. 1–12.

[2] M. Ben-Artzi and S. Klainerman: *Decay and regularity for the Schrödinger equation*, J. Anal. Math. **58** (1992), 25–37.

[3] M. Blair, H. Smith and C. D. Sogge: *Strichartz estimates for the wave equation on manifolds with boundary*, arXiv:0805.4733.

[4] N. Burq: *Global Strichartz estimates for nontrapping geometries: About an article by H. Smith and C. Sogge*, Comm. Partial Differential Equations **28** (2003), 1675–1683.

[5] N. Burq, G. Lebeau and F. Planchon: *Global existence for energy critical waves in 3-D domains*, J. Amer. Math. Soc. **21** (2008), 831–845.

[6] N. Burq and F. Planchon: *Global existence for energy critical waves in 3-d domains : Neumann boundary conditions*, arXiv:0711.0275.

[7] M. Christ and A. Kiselev: *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), 409–425.

[8] Y. Du, J. Metcalfe, C. D. Sogge and Y. Zhou: *Concerning the Strauss conjecture and almost global existence for nonlinear Dirichlet-wave equations in* 4*-dimensions*, Comm. Partial Differential Equations **33** (2008), 1487–1506.

[9] Y. Du and Y. Zhou: *The life span for nonlinear wave equation outside of star-shaped obstacle in three space dimensions*, Comm. Partial Differential Equations **33** (2008), 1455–1486.

[10] D. Fang and C. Wang: *Weighted Strichartz Estimates with Angular Regularity and their Applications*, arXiv:0802.0058.

[11] V. Georgiev, H. Lindblad, and C. D. Sogge: *Weighted Strichartz estimates and global existence for semilinear wave equations*, Amer. J. Math. **119** (1997), 1291–1319.

[12] R. T. Glassey: *Existence in the large for* $\Box u = F(u)$ *in two dimensions*, Math. Z. **178** (1981), 233–261.

[13] K. Hidano: *Morawetz-Strichartz estimates for spherically symmetric solutions to wave equations and applications to semi-linear Cauchy problems*, Differential Integral Equations **20** (2007), 735–754.

[14] K. Hidano: *Small solutions to semi-linear wave equations with radial data of critical regularity*, Rev. Mat. Iberoamericana, to appear.

[15] T. Hoshiro: *On weighted L^2 estimates of solutions to wave equations*, J. Anal. Math. **72** (1997), 127–140.

[16] F. John: *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–265

[17] M. Keel, H. F. Smith, and C. D. Sogge: *Almost global existence for some semilinear wave equations*, J. Anal. Math. **87** (2002), 265-279.

[18] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

[19] H. A. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$*, Trans. Amer. Math. Soc. **192** (1974), 1–21.

[20] T. T. Li and Y. Zhou: *A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions*, Indiana Univ. Math. J. **44** (1995), 1207–1248.

[21] H. Lindblad and C. D. Sogge: *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130** (1995), 357–426.

[22] H. Lindblad and C. D. Sogge: *Long-time existence for small amplitude semilinear wave equations*, Amer. J. Math. **118** (1996), 1047–1135.

[23] R. B. Melrose: *Singularities and energy decay in acoustical scattering*, Duke Math. J. **46** (1979), 43–59.

[24] R. B. Melrose and J. Sjöstrand: *Singularities of boundary value problems. I*, Comm. Pure Appl. Math. **31** (1978), 593–617.

[25] J. Metcalfe: *Global Strichartz estimates for solutions to the wave equation exterior to a convex obstacle*, Trans. Amer. Math. Soc. **356**, (2004), 4839–4855.

[26] C. S. Morawetz: *Decay for solutions of the exterior problem for the wave equation*, Comm. Pure and Appl. Math. **28** (1975), 229–264.

[27] C. S. Morawetz, J. Ralston and W. Strauss: *Decay of solutions of the wave equation outside nontrapping obstacles*, Comm. Pure Appl. Math. **30** (1977), 87–133.

[28] P. D. Lax and R. S. Philips: Scattering Theory (Revised Edition), Academic Press Inc., 1989.

[29] J. Ralston: *Note on the decay of acoustic waves*, Duke Math. J. **46** (1979), 799-804.

[30] Y. Shibata and Y. Tsutsumi: *Global existence theorem for nonlinear wave equation in exterior domain*, Lecture Notes in Num. Appl. Anal., Vol. 6 (1983), 155-196.

[31] T. C. Sideris: *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations **52** (1984), 378–406.

[32] H. F. Smith and C. D. Sogge: *On the critical semilinear wave equation outside convex obstacles*, J. Amer. Math. Soc. **8** (1995), 879–916.

[33] H. F. Smith and C. D. Sogge: *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations **25**, (2000), 2171–2183.

[34] H. F. Smith and C. D. Sogge: *On the L^p norm of spectral clusters for compact manifolds with boundary*, Acta Math. **198**, (2007), 107–153.

[35] C. D. Sogge: Lectures on nonlinear wave equations, International Press, Boston, MA 1995.

[36] C. D. Sogge: Lectures on nonlinear wave equations, 2nd edition, International Press, Boston, MA, 2008.

[37] D. Tataru: *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. **353** (2001), 795–807.

[38] M. Taylor: *Grazing rays and reflection of singularities of solutions to wave equations*, Comm. Pure Appl. Math. **29** (1976), 1–38.

[39] B. R. Vainberg: *The short-wave asymptotic behavior of the solutions of stationary problems, and the asymptotic behavior as $t \to \infty$ of the solutions of nonstationary problems*, Russian Math. Surveys **30** (1975), 1–58.

[40] Y. Zhou: *Cauchy problem for semilinear wave equations with small data in four space dimensions*, J. Partial Differential Equations **8** (1995), 135–144.

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