PARAMETRIX FOR A SEMICLASSICAL SUBELLIPTIC OPERATOR

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Abstract. We demonstrate a parametrix construction, together with associated pseudodifferential operator calculus, for an operator of sum-of-squares type with semiclassical parameter. The form of operator we consider includes the generator of kinetic Brownian motion on the cosphere bundle of a Riemannian manifold. Regularity estimates in semiclassical Sobolev spaces are proven by establishing mapping properties for the parametrix.

1. introduction

We deal in this paper with a class of second order, subelliptic partial differential operators of the following sum-of-squares form

$$P_h = X_0 - h\sum_{j=1}^{d} X_j^2 - h\sum_{j=1}^{d} c_j X_j\,, \qquad h\in(0,1], \tag{1.1}$$

where the X_j for $0\le j\le d$ are smooth vector fields, the c_j are smooth functions, and $h>0$ is considered as a semiclassical parameter. We work in $2d+1$ dimensions, either on a compact manifold or an open subset of $\mathbb{R}^{2d+1}$, and make the following assumptions throughout this paper.

Assumption 1.

- The collection of $2d+1$ vectors $\{X_0, X_1, \dots X_d, [X_0, X_1], \dots, [X_0, X_d]\}$ spans the tangent space at each base point.
- The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).

For each $h>0$ the operator P_h is subelliptic by a result of Hörmander [Hör67], and by the work of Rothschild-Stein [RS76] the operator P_h controls 2/3-derivatives in the Sobolev space sense. In the semiclassical setting it is natural to work with a semiclassical notion of Sobolev spaces; we refer to the text of Zworski [Zwo12] for a treatment of semiclassical analysis. The question of interest in this paper is the dependence on h of the various constants in a priori inequalities for P_h, both in L^2 and semiclassical Sobolev spaces.

Our work is motivated by the paper [Dro17] of Alexis Drouot, which studied such an operator on the cosphere bundle $S^*(M)$ of a $d+1$ dimensional Riemannian manifold M. The paper [Dro17] considers the operator $P_h = H + h\Delta_{\mathbb{S}}$, with H the generator of the Hamiltonian/geodesic flow, and $\Delta_{\mathbb{S}}$ the non-negative Laplace-Beltrami operator along the fibers of the cosphere bundle. In local coordinate charts this operator can be represented in the form (1.1), where the X_j are any

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local orthonormal frame for the tangent space of the fibers of $S^*(M)$. In [Dro17] it is shown that, if M is negatively curved, then as $h \to 0$ the eigenvalues of $-iP_h$ converge to the Pollicott-Ruelle resonances of M. The analogous result was proven by Dyatlov-Zworski [DZ15] for $P_h = H + h\Delta$, where Δ is the Laplacian on $S^*(M)$. The interest in taking $P_h = H + h\Delta_{\mathbb{S}}$ is that this operator generates what is known as kinetic Brownian motion on M. For a treatment of this process we refer to Franchi-Le Jan [FLJ07], Grothaus-Stilgenbauer [GS13], Angst-Bailleul-Tardif [ABT15], and Li [Li16].

A key step in the proof of convergence in [Dro17] was controlling the subelliptic estimates for P_h as $h \to 0$. This was done through commutator methods, analogous to the work of [Hör67]. Our approach is more similar to that of [RS76], in that we use an approximation to the operator at each point by a model nilpotent group, and construct a parametrix from the inverse of the model operator on that group. Estimates are then obtained from mapping properties for the parametrix. We emphasize that the estimates we prove are the same as in [Dro17], with an occasional improvement in the remainder terms. The aim here is to obtain a finer microlocal understanding of the parametrix. We obtain a parametrix valid on the region $h\Delta \geq 1$, strictly larger than the semiclassical region $h^2\Delta \geq 1$. The restriction $h\Delta \geq 1$ arises from the largest region of phase space on which the uncertainty principle holds for the parametrix.

We now mention a few of the features encountered in the parametrix construction for an operator P_h of the form (1.1). The quantization of symbols is naturally carried out using exponential coordinates with respect to an extension of $\{X_j\}_{j=0}^d$ to a frame $\{X_j\}_{j=0}^{2d+1}$. We will require that

Assumption 2.

- If $1 \leq i \leq d$, then $[X_0, X_i] - 2X_{i+d} \in \operatorname{span}(X_0, \dots, X_d)$,

This can of course be arranged by setting $X_{i+d} = 2[X_0, X_i]$. In the model nilpotent Lie group setting where all other commutators vanish, there is a natural nonisotropic dilation structure using powers $(2,1,3)$. Precisely, we split $\eta \in \mathbb{R}^{2d+1} = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ into (η_0, η', η''), and similarly use X' as abbreviation for the collection $(X_1, \dots, X_d)$, and $X'' = (X_{d+1}, \dots, X_{2d})$. Then the dilation that respects the fundamental solution for the model of P_0 is

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta'').$$

From the semiclassical point of view it is more natural to consider $hP_h = hX_0 + \sum_{j=1}^d (hX_j)^2$, and quantize symbols in terms of $h\eta$. This leads to placing an extra factor of h in the variable η'' dual to X_j for $d+1 \leq j \leq 2d$, since $[hX_0, hX_j] \sim h^2 X_{j+d}$.

We now summarize the main result of this paper, leaving details to be expanded upon in later sections. For simplicity consider an open set $U \subset \mathbb{R}^{2d+1}$. For a multi-index $\alpha \in \mathbb{N}^{2d+1}$, let

$$\operatorname{order}(\alpha) = 2\alpha_0 + |\alpha'| + 3|\alpha''|.$$

We use $\exp_x(y)$ to denote the time 1 flow of x along $\sum_{j=0}^{2d} y_j X_j$.

Proposition 1.1. *Given $\rho(x) \in C_c^\infty(U)$, there is $\chi_0 \in C_c^\infty(\mathbb{R}^{2d+1})$, and an h-dependent family of symbols $a(x,\eta)$ satisfying*

$$\left|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)\right| \leq C_{\alpha,\beta}\, h\Big(h^{\frac12} + |\eta_0|^{\frac12} + |\eta'| + |\eta''|^{\frac13}\Big)^{-2-\operatorname{order}(\alpha)}$$

with $C_{\alpha,\beta}$ *independent of* $h \in (0, 1]$*, so that the operator* $a_h(x, hD)$ *defined by*

$$a_h(x, hD) = \frac{1}{(2\pi)^{2d+1}} \int e^{-i\langle y,\eta\rangle}\, a(x, h\eta_0, h\eta', h^2\eta'')\, f(\exp_x(y))\, \chi_0(y)\, dy\, d\eta$$

satisfies

$$a_h(x, hD) \circ P_h = \rho(x) + r_h(x, hD),$$

where $r_h(x, hD)$ *is an operator that satisifies the following with* C_{p_1,p_2} *independent of* $h \in (0, 1]$*, for any polynomials* $p_j(\eta)$ *on* $\mathbb{R}^{2d+1}$*,*

$$\big\|p_1(X_0, h^{\frac12}X', h^{\frac12}X'') \circ r_h(x, hD) \circ p_2(X_0, h^{\frac12}X', h^{\frac12}X'')f\big\|_{L^2} \le C_{p_1,p_2}\|f\|_{L^2}. \tag{1.2}$$

For example, one can take p_1 or p_2 to yield the operator $(1 + X_0^*X_0)^{N_1}(1 + h\Delta)^{N_2}$, where Δ is the Laplacian on $\mathbb{R}^{2d+1}$. These bounds roughly say that the parametrix inverts P_h on the region $\{\Delta \ge h^{-1}\} \cup \{|X_0| \ge 1\}$. In particular the remainder term r_h will be of order h^∞ if the solution is localized to a region where $\Delta \ge h^{-1-\epsilon}$ for some $\epsilon > 0$.

We remark that in the calculus developed here P_h is of order 2, in distinction with the standard semiclassical calculus where hP_h is order 2. This is related to our working on the region $|\eta| \ge h^{\frac12}$, and the subelliptic nature of P_h. Symbols of order j are weighted by a factor $h^{-j/2}$, so that symbols of negative order (but not necessarily their derivatives) remain bounded as $h \to 0$. With this accounting X_0 is an operator of order 2, $h^{\frac12}X_j$ is of order 1 for $1 \le j \le d$, and $h^{\frac12}X_j$ is of order 3 for $d + 1 \le j \le 2d$.

Together with the composition calculus, pseudolocality arguments, and L^2 mapping bounds for operators, we deduce the regularity results on $S^*(M)$ for P_h that were established in [Dro17]. These are stated in Theorems 6.3 and 6.4.

The outline of this paper is as follows. In Section 2 we introduce a model operator of P_0 and P_h on a step-2 nilpotent group, and discuss composition of symbols in this setting. In Section 3 we discuss the degree to which the model operator, attached to M by exponential coordinates, approximates P_h. Careful estimates of the Taylor expansion of vector fields and exponential coordinates are needed to obtain uniform estimates as $h \to 0$. In Section 4 we prove that operators of the form $a_h(x, hD)$ form an algebra under composition, and that the symbol of the composition of two operators agree at a point x, modulo an operator of one lower order, with composition on the attached model domain above x. This allows for iteration of symbols and construction of parametrices from a suitable inverse for the model operator on the nilpotent Lie group. In Section 5 we establish L^2 boundedness of order 0 operators in local coordinates, using a nonisotropic Littlewood-Paley decomposition of the operator and the Cotlar-Stein lemma. Finally, in Section 6 we establish the main regularity estimates for P_h in h-Sobolev spaces, leading to the proof of the bounds in [Dro17].

2. Operators on model domains

In this section we work with translation invariant operators associated to a nilpotent Lie group structure on $\mathbb{R}^{2d+1}$. We use notation $y = (y_0, y', y'') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, and dual variables $\eta = (\eta_0, \eta', \eta'')$. The dilation structure is given by

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta''), \qquad \delta_{r^{-1}}(y) = (r^{-2}y_0, r^{-1}y', r^{-3}y'').$$

We introduce a corresponding non-isotropic homogeneous weight $m \in C^\infty(\mathbb{R}^{2d+1}\backslash\{0\})$,

$$m(\eta) = \Big(|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4\Big)^{\frac{1}{12}},$$

so that $m(\delta_r(\eta)) = rm(\eta)$, and $3^{-\frac{12}{5}} \le m(\eta) \le 1$ when $|\eta| = 1$.

The order of a multi-index α is defined to be

$$\mathrm{order}(\alpha) = 2\alpha_0 + \alpha_1 + \cdots + \alpha_d + 3\alpha_{d+1} + \cdots + 3\alpha_{2d} = 2\alpha_0 + |\alpha'| + 3|\alpha''|,$$

and we define the order of a monomial differential operator by

$$\mathrm{order}\Big(y^\beta \partial_y^\alpha\Big) = \mathrm{order}(\alpha) - \mathrm{order}(\beta). \tag{2.1}$$

We work with the following frame of vector fields on $\mathbb{R}^{2d+1}$:

- $Y_0 = \partial_0 - \sum_{j=1}^d y_j \partial_{j+d}$
- $Y_j = \partial_j + y_0 \partial_{j+d}$ for $1 \le j \le d$,
- $Y_j = \partial_j$ for $j \ge d+1$.

Observe that $\mathrm{order}(Y_j) = \mathrm{order}(\partial_j)$, and that

$$[Y_0, Y_j] = 2Y_{j+d} \quad \text{if} \quad 1 \le j \le d,$$

and all other commutators are equal to 0. The Y_j form a nilpotent (step 2) Lie algebra, which is associated to the nilpotent Lie group structure on $\mathbb{R}^{2d+1}$ with product

$$y \times w = (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y').$$

We consider in this section left invariant pseudodifferential operators on $\mathbb{R}^{2d+1}$ associated to the model vector fields Y_j, which we quantize using the exponential map associated to the Y_j. This map, and the corresponding exponential coordinates at base point y, are given by

$$\begin{aligned} \bar{\exp}_y(w) &= (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y'), \\ \bar{\Theta}_y(z) &= (z_0 - y_0, z' - y', z'' - y'' - y_0 z' + z_0 y'). \end{aligned} \tag{2.2}$$

A multiplier $a(\eta) \in C_c^\infty(\mathbb{R}^{2d+1})$ is then associated to the Schwartz kernel

$$K_a(y,z) = \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{2d+1}} e^{-i\langle \bar{\Theta}_y(z), \eta\rangle}\, a(\eta)\, d\eta,$$

and hence $a(\xi) = \int K(0,z) e^{i\langle z,\xi\rangle}\, dz$. The composition rule for multipliers is given by

$$(a \sharp b)(\xi) = \frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y, \eta-\zeta\rangle + i\langle y_0 z' - z_0 y', \zeta''\rangle + i\langle z, \xi - \zeta\rangle}\, a(\eta)\, b(\zeta)\, dz\, d\zeta\, dy\, d\eta.$$

The integral over $dz''\, d\zeta''\, dy''\, d\eta''$ fixes $\eta'' = \zeta'' = \xi''$, and we get

$$\frac{1}{(2\pi)^{2d+2}} \int e^{iy_0(\eta_0 - \xi_0) + i\langle y', \eta' - \xi'\rangle + iz_0(\zeta_0 - \eta_0) + i\langle z', \zeta' - \eta'\rangle + i\langle y_0 z' - z_0 y', \xi''\rangle} a(\eta_0, \eta', \xi'') b(\zeta_0, \zeta', \xi'')$$
$$d\eta_0\, d\eta'\, d\zeta_0\, d\zeta'\, dy_0\, dy'\, dz_0\, dz',$$

which simplifies to give

$$(a\sharp b)(\xi) = \frac{1}{(2\pi)^{d+1}} \int e^{iz_0\zeta_0 + i\langle z',\zeta'\rangle} a(\xi_0 - \langle z',\xi''\rangle, \xi' + z_0\xi'', \xi'') b(\xi_0+\zeta_0, \xi'+\zeta', \xi'')\, d\zeta_0\, d\zeta'\, dz_0\, dz'.$$

Observe that model composition commutes with the dilations $\delta_{r,h}$, in that

$$(a\sharp b)\circ\delta_{r,h} = (a\circ\delta_{r,h})\sharp(b\circ\delta_{r,h}). \tag{2.3}$$

Theorem 2.1. *Suppose that, for some $r_0 > 0$, the symbols a and b satisfy*

$$\left|\partial_\xi^\alpha a(\xi)\right| \le C_\alpha \big(r_0 + m(\xi)\big)^{n-\mathrm{order}(\alpha)},$$
$$\left|\partial_\xi^\alpha b(\xi)\right| \le C_\alpha \big(r_0 + m(\xi)\big)^{n'-\mathrm{order}(\alpha)}.$$

Then $a\sharp b$ is well defined as an oscillatory integral, and for all α,

$$\left|\partial_\xi^\alpha (a\sharp b)(\xi)\right| \le C'_\alpha \big(r_0 + m(\xi)\big)^{n+n'-\mathrm{order}(\alpha)},$$

where each C'_α depends on n, n' and a finite number of the C_α, but is independent of r_0.

Proof. By composing with $\delta_{r_0,1}$ we may assume $r_0 = 1$, which shows the independence of the C'_α from r_0. Applying ∂_ξ^α to the expression for $a\sharp b$ decreases the combined order of the symbols in the integrand by order(α), so it is sufficient to show $|(a\sharp b)(\xi)| \le C\,(1+m(\xi))^{n+n'}$, for C depending on a finite number of the C_α. We use the following inequality,

$$\frac{1+m(\zeta_0,\zeta',\xi'')}{1+m(\eta_0,\eta',\xi'')} \le C\left(1 + \frac{|\zeta_0-\eta_0|}{\langle\xi''\rangle^{\frac13}} + \frac{|\zeta'-\eta'|}{\langle\xi''\rangle^{\frac23}}\right) \tag{2.4}$$

which follows by writing the ratio on the left as comparable to

$$\left(1 + \frac{|\zeta_0|}{\langle\xi''\rangle^{\frac13}} + \left(\frac{|\zeta'|}{\langle\xi''\rangle^{\frac23}}\right)^{\frac12}\right)\Big/\left(1 + \frac{|\eta_0|}{\langle\xi''\rangle^{\frac13}} + \left(\frac{|\eta'|}{\langle\xi''\rangle^{\frac23}}\right)^{\frac12}\right)$$

and applying Peetre's inequality. Integration by parts with respect to the operators

$$L_1 = \frac{1 - \langle\xi''\rangle^{\frac23}\partial_{\zeta_0}^2 - \langle\xi''\rangle^{\frac43}\partial_{\zeta'}^2}{1+\langle\xi''\rangle^{\frac23}|z_0|^2 + \langle\xi''\rangle^{\frac43}|z'|^2}, \qquad L_2 = \frac{1 - \langle\xi''\rangle^{-\frac23}\partial_{z_0}^2 - \langle\xi''\rangle^{-\frac43}\partial_{z'}^2}{1+\langle\xi''\rangle^{-\frac23}|\zeta_0|^2 + \langle\xi''\rangle^{-\frac43}|\zeta'|^2}$$

produces an integral dominated by a constant C' as in the lemma multiplied by

$$\int \frac{\big(1+m(\xi_0-\langle z',\xi''\rangle, \xi'+z_0\xi'', \xi'')\big)^n \big(1+m(\xi_0+\zeta_0, \xi'+\zeta', \xi'')\big)^{n'}}{\big(1+\langle\xi''\rangle^{\frac23}|z_0|^2 + \langle\xi''\rangle^{\frac43}|z'|^2 + \langle\xi''\rangle^{-\frac23}|\zeta_0|^2 + \langle\xi''\rangle^{-\frac43}|\zeta'|^2\big)^{|n|+|n'|+d+1}}\, d\zeta_0\, d\zeta'\, dz_0\, dz'. \tag{2.5}$$

By (2.4) this integral is bounded by $C_{n,n'}\big(1+m(\xi)\big)^{n+n'}$. □

Suppose now that a and b satisfy the conditions of Theorem 2.1 with $r_0 = 0$. For each $r_0 > 0$ the truncated symbol $\big(1-\chi(r_0^{-1}m(\xi))\big)a(\xi)$ satisfies the conditions of the lemma for r_0, if $\chi \in C_c^\infty((-2,2))$ with $\chi(r) = 1$ for $r \le 1$. Since $m(\xi) \ge |\xi''|^{\frac13}$, for $\xi'' \ne 0$ the integral defining $a\sharp b$ is the same if we replace a and b by their truncations to $m(\xi) \ge r_0$ for r_0 suitably small depending on ξ''. We thus obtain for $\xi'' \ne 0$ that

$$\left|\partial_\xi^\alpha (a\sharp b)(\xi)\right| \le C'_\alpha m(\xi)^{n+n'-\mathrm{order}(\alpha)}.$$

Using scaling and dominated convergence, one can show that if $(\xi_0,\xi')\neq 0$ then as $\xi''\to 0$ we have $(a\sharp b)(\xi)\to a(\xi_0,\xi',0)b(\xi_0,\xi',0)$. Together with differentiation under the integral this shows that $a\sharp b\in C^\infty(\mathbb{R}^{2d+1}\backslash\{0\})$. We conclude, recalling also (2.3),

Corollary 2.2. *Suppose that the symbols a and b belong to $C^\infty(\mathbb{R}^{2d+1}\backslash\{0\})$ and satisfy*

$$\begin{aligned}|\partial_\xi^\alpha a(\xi)| &\leq C_\alpha m(\xi)^{n-\mathrm{order}(\alpha)},\\ |\partial_\xi^\alpha b(\xi)| &\leq C_\alpha m(\xi)^{n'-\mathrm{order}(\alpha)}.\end{aligned}$$

Then $a\sharp b$, defined as the limit as $r\to 0$ of truncation to $m(\xi)>r$, satisfies for all α

$$|\partial_\xi^\alpha (a\sharp b)(\xi)|\leq C'_\alpha m(\xi)^{n+n'-\mathrm{order}(\alpha)},$$

where each C'_α depends on n, n' and a finite number of the C_α. Furthermore, if a and b are homogeneous under δ_r of degree n and n', then $a\sharp b$ is homogeneous of degree $n+n'$.

We also observe here the following result.

Lemma 2.3. *If a and b satisfy the conditions of Corollary 2.2, then*

$$\Big((1-\chi(r^{-1}m(\cdot)))a\Big)\sharp\Big((1-\chi(r^{-1}m(\cdot)))b\Big)(\xi)=(1-\chi(r^{-1}m(\xi)))(a\sharp b)(\xi)+r^{n+n'}c(r,\delta_{r^{-1}}(\xi)),$$

where $c(r,\cdot)$ is a family of Schwartz functions with seminorms uniformly bounded over $r>0$.

Proof. We dilate ξ by δ_r, and note that $r^{-n}a(\delta_r(\xi))$ satisfies the estimates of Corollary 2.2, with constants C_α independent of r, similarly for $r^{-n'}b(\delta_r(\xi))$. Thus it suffices to prove the result for $r=1$, together with showing that the Schwartz norm of $c(1,\xi)$ depends only on the constants C_α for a and b. Since the terms involved have bounded C^∞ norm over $|\xi|\leq 8$, it suffices to show that, with $\chi(\xi)=\chi(m(\xi))\in C_c^\infty(|\xi|\leq 8)$,

$$a\sharp b-((1-\chi)a)\sharp((1-\chi)b)=(\chi a)\sharp((1-\chi)b)+a\sharp(\chi b)$$

satisfies Schwartz bounds on the set $|\xi|>24$. Consider $a\sharp(\chi b)$. Since χb is supported where $m(\xi)\leq 2$, and its derivatives agree with derivatives of b for $m(\xi)\leq 1$, it satisfies

$$|\partial_\xi^\alpha(\chi(\xi)b(\xi))|\leq C_{\alpha,N}\,m(\xi)^{-N-\mathrm{order}(\alpha)}\quad\text{for all } N\geq -n'.$$

Thus Corollary 2.2 yields Schwartz bounds on $a\sharp(\chi b)$ for $|\xi|$ bounded away from 0. $\square$

The left invariant differential operator $Y_0-\sum_{j=1}^d Y_j^2$ is subelliptic, and by Folland [Fol75] admits a unique homogeneous fundamental solution $K(y)\in C^\infty(\mathbb{R}^{2d+1})$,

$$\Big(Y_0-\sum_{j=1}^d Y_j^2\Big)K(y)=\delta(y),\qquad K(\delta_{r^{-1}}(y))=r^{(2+4d)-2}K(y).$$

We let $q(\eta)=\widehat{K}$. Then the operator

$$T_qf(y)=\frac{1}{(2\pi)^{2d+1}}\int_{\mathbb{R}^{2d+1}}e^{-i\langle\bar\Theta_y(z),\eta\rangle}\,q(\eta)\,f(z)\,dz\,d\eta$$

is a left and right inverse for $Y_0-\sum_{j=1}^d Y_j^2$ on the space of Schwartz functions.

We next consider the semiclassical subelliptic operator $hY_0 - \sum_{j=1}^d h^2 Y_j^2$. This is naturally associated to dilating y_0 and y' by h, and y'' by h^2, in that

$$\Big(Y_0 - \sum_{j=1}^d Y_j^2\Big)\Big(f(hy_0, hy', h^2 y'')\Big) = \Big(hY_0 f - \sum_{j=1}^d h^2 Y_j^2 f\Big)(hy_0, hy', h^2 y'').$$

Consequently, if we introduce the operation on symbols

$$a_h(\eta) = a(\eta_0, \eta', h\eta''), \tag{2.6}$$

then the inverse for $hY_0 - \sum_{j=1}^d h^2 Y_j^2$ is given by the semiclassical quantization of q_h,

$$\begin{aligned} q_h(hD)f(y) &\equiv \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle \bar{\Theta}_y(z), \zeta\rangle/h}\, q_h(\eta)\, f(z)\, dz\, d\zeta \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle z, \zeta\rangle}\, q_h(\eta)\, f(\bar{\exp}_y(hz))\, dz\, d\zeta \end{aligned}$$

3. Approximation by the model domain

Recall that we consider a spanning collection $\{X_0, X_1, \ldots, X_{2d}\}$ of vector fields on an open subset U of $\mathbb{R}^{2d+1}$, satisfying the following conditions:

- The collection $\{X_1, \ldots, X_d\}$ is involutive (closed under commutation of vector fields).
- If $1 \le i \le d$, then $[X_0, X_i] - 2X_{i+d} \in \operatorname{span}(X_0, \ldots, X_d)$.

We will use $x, \tilde{x}$ to denote variables in U, and y, z denote variables in $\mathbb{R}^{2d+1}$.

Let $\exp_x(y)$ be exponential coordinates with base point x in the frame $\{X_0, \ldots, X_{2d}\}$. That is, $\exp_x(y) = \gamma(1)$ where $\gamma(0) = x$ and $\gamma'(t) = \sum_{j=0}^{2d} y_j X_j(\gamma(t))$. Define exponential coordinates $\Theta_x(\tilde{x})$ as the local inverse for $\tilde{x}$ in a neighborhood of x:

$$\Theta_x(\exp_x(y)) = y, \qquad \exp_x(\Theta_x(\tilde{x})) = \tilde{x}.$$

Lemma 3.1. *For $0 \le j \le 2d$, we can write*

$$(X_j f)(\exp_x(y)) = Y_j\Big(f(\exp_x(y))\Big) + R_j(x, y, \partial_y) f(\exp_x(y)),$$

where $\operatorname{order}(R_j) < \operatorname{order}(Y_j)$, *in the sense that the Taylor expansion*

$$R_j(x, y, \partial_y) = \sum_{\alpha, k} c_{j,\alpha,k}(x)\, y^\alpha \partial_k$$

includes only terms with $\operatorname{order}(y^\alpha \partial_k) < \operatorname{order}(Y_j)$.

Additionally, $c_{0,\alpha,k}(x) \equiv 0$ unless there is at least one factor of y_j with $j \ge 1$ occuring in y^α.

Proof. Any term $y^\alpha \partial_y^\beta$ with $|\alpha| > 2$ is of order ≤ 0, so we need examine the Taylor expansion of X_j in exponential coordinates only to second power in y. Additionally, $\operatorname{order}(y_i y_j \partial_k) \le 1$, and equals 1 only if $1 \le i, j \le d$ and $k \ge d+1$. To see that such a term cannot arise in the expansion of X_j for $1 \le j \le d$, which are the only X_j of order 1, we use involutivity of $\{X_1, \ldots, X_d\}$ to see we can write $X_j = \sum_{j=1}^d c_j(x, y)\partial_j$ at all points in the subspace $y_0 = y'' = 0$.

Thus, we need show that in the expansion of R_j about $y=0$ the terms linear in y are of order strictly less than order(Y_j). For $j \geq d+1$ this is immediate, since $X_j = \partial_j = Y_j$ at $y=0$, and any vector field that vanishes at 0 is of order at most 2. For $0 \leq j \leq d$ we expand

$$X_j = \partial_j + \sum_{i,k} c_{jik}(x)\, y_i \partial_k + \mathcal{O}(y^2)\partial_y.$$

From the relation $\sum y_j X_j = \sum y_j \partial_j$, we deduce

$$c_{ijk} = -c_{jik},$$

and from $[X_0, X_j] - 2X_{j+d} \in \operatorname{span}(X_0, \ldots, X_d)$, we deduce for $j = 1, \ldots d$ that

$$c_{j0k} = \begin{cases} 1, & k = j+d, \\ 0, & k > d \ \text{ and } \ k \neq j+d. \end{cases}$$

Since order$(y_i \partial_k) < 2$ unless $k > d$, we deduce order$\bigl(R_0(x, y, \partial_y)\bigr) < 2$.

By the involutivity assumption, if $1 \leq i, j \leq d$ then $c_{jik} = 0$ unless also $1 \leq k \leq d$, in which case order$(y_i \partial_k) = 0$. And if $i > d$ then order$(y_i \partial_k) \leq 0$ for all k. So if $1 \leq j \leq d$ then all terms $c_{jik}\, y_i \partial_k$ for $i \neq 0$ have order ≤ 0, and since $c_{j0k} = \delta_{k,j+d}$ we conclude order$\bigl(R_j(x, y, \partial_y)\bigr) \leq 0$ if $1 \leq j \leq d$.

To conclude the lemma, we note that if $y' = y'' = 0$ then $X_0 = \partial_{y_0}$, hence $R_0 = 0$. $\square$

For $x \in U$, and y, z in a neighborhood of 0 in $\mathbb{R}^{2d+1}$, we introduce the functions

$$\Theta(x, y, z) = \Theta_{\exp_x(y)}\bigl(\exp_x(z)\bigr), \quad \tilde{\Theta}(x, y, w) = \Theta_x\bigl(\exp_{\exp_x(y)}(w)\bigr). \tag{3.1}$$

where we recall $\Theta_x(\tilde{x})$ denotes exponential coordinates in X_j centered at x. For fixed x and y these are inverse functions of each other on their domains,

$$z = \tilde{\Theta}(x, y, w) \quad \Leftrightarrow \quad w = \Theta(x, y, z).$$

To invert in the y variable we note that $v = \tilde{\Theta}(x, y, w)$ implies $y = \tilde{\Theta}(x, v, -w)$.

Observe that $\Theta(x, y, z) = -\Theta(x, z, y)$, and $\Theta(x, y, z) = z - y + \mathcal{O}(y, z)^2$. For more precise estimates on Θ and $\tilde{\Theta}$ we consider their Taylor expansions in exponential coordinates at x. We first assign a notion of order to a smooth function $f(x, y, z)$. Consistent with (2.1), we make the following definition.

Definition 3.1. For a smooth function $f(x, y, z)$ defined on an open subset of $U \times \mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1}$ containing $U \times \{0, 0\}$, we say that order$(f) < -j$ if for all $x \in U$

$$\partial_y^\alpha \partial_z^\beta f(x, 0, 0) = 0 \quad \text{for all } \alpha, \beta : \operatorname{order}(\alpha + \beta) \leq j.$$

Equivalently, the Taylor expansion of f in y, z about $y = z = 0$ contains only monomials $y^\alpha z^\beta$ with order$(\alpha + \beta) > j$.

Recalling the definition (2.2) of $\bar{\Theta}_y(z)$, we have the following.

Lemma 3.2. *We have* $\Theta(x, y, z) = \bar{\Theta}_y(z) + R(x, y, z)$, *where* order$(R_j) <$ order(y_j) *for each* j. *Similarly,* $\tilde{\Theta}(x, y, z) = \bar{\Theta}_{-y}(w) + \tilde{R}(x, y, z)$, *where* order$(\tilde{R}_j) <$ order(y_j) *for each* j.

Proof. We work in exponential coordinates centered at x, and use Lemma 3.1 to consider X_j as a vector field in y. Then for a given y the Taylor expansion of $z = \tilde{\Theta}(x,y,w)$ in terms of w is

$$z_j = y_j + (w \cdot X)_j(y) + \sum_{k=1}^{\infty} \frac{1}{(k+1)!}(w \cdot X)^k (w \cdot X)_j(y), \tag{3.2}$$

where $w \cdot X = \sum_{i=0}^{2d} w_i X_i$ is a vector field acting on the y variable, and $(w \cdot X)_j(y)$ is its ∂_j coefficient as a function of y. It is seen from Lemma 3.1 that $w \cdot X$ does not increase the order of a function $f(x,y,w)$, and $w \cdot X - w \cdot Y$ decreases the order of $f(x,y,w)$ by at least 1. Also, as functions of (y,w)

$$\mathrm{order}\big((w \cdot Y)_j(y)\big) = \mathrm{order}(y_j), \qquad \mathrm{order}\big((w \cdot X)_j(y) - (w \cdot Y)_j(y)\big) < \mathrm{order}(y_j).$$

Thus, if we replace $w \cdot X$ by $w \cdot Y$ in the expansion (3.2) then the right hand side is changed by terms of strictly lower order than y. It follows that we can write

$$(z_0, z', z'') = (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y') + (\tilde{R}_0, \tilde{R}', \tilde{R}''), \tag{3.3}$$

where $\mathrm{order}(\tilde{R}_0) < -2$, $\mathrm{order}(\tilde{R}') < -1$, and $\mathrm{order}(\tilde{R}'') < -3$ as functions of (y,w).

We now express $w = w(y,z)$ and use (3.3) to write

$$w = \bar{\Theta}_y(z) - (\tilde{R}_0, \tilde{R}', -y_0 \tilde{R}' + y' \tilde{R}_0 + \tilde{R}'') \tag{3.4}$$

where $\tilde{R} = \tilde{R}(y, w(y,z))$. Since $w = z - y$ plus quadratic terms in (y,z), we see that $R(x,y,z)$ has no linear terms in y or z, and also that $\mathrm{order}(w_0) \le -2$ and $\mathrm{order}(w') \le -1$, since quadratic terms are of order at most -2.

It suffices now to show that $\mathrm{order}(w''(y,z)) \le -3$, since together with the preceding we have $\mathrm{order}(w_j(y,z)) \le \mathrm{order}(y_j)$ for all j, from which it follows that $\mathrm{order}(\tilde{R}_j(y, w(y,z))) < \mathrm{order}(y_j)$. We know $\mathrm{order}(w''(y,z)) \le -2$ since quadratic terms are order ≤ -2. On the other hand, since $\mathrm{order}(\tilde{R}'') \le -4$ as a function of (y,w), and $\mathrm{order}(w_j(y,z)) \le \mathrm{order}(y_j)$ for $j \le d$, it is easy to see by examining possible terms in $\tilde{R}''$ that $\mathrm{order}(\tilde{R}'') \le -3$ as a function of (y,z), concluding the proof. □

We make a few important additional observations about the terms that can occur in the Taylor expansion of Θ and $\tilde{\Theta}$ about $y = z = 0$, respectively $y = w = 0$. First, we have

$$\Theta(x,y,z) = z_0 - y_0 \ \text{ if } \ y' = z' = y'' = z'' = 0,$$
$$\tilde{\Theta}(x,y,w) = y_0 + w_0 \ \text{ if } \ y' = w' = y'' = w'' = 0.$$

Consequently, every nonvanishing term in the Taylor expansion of $R(x,y,z)$ must include a factor of either y', z', y'', or z''. Similarly, every nonvanishing term in the Taylor expansion of $\tilde{R}(x,y,z)$ must include factor of either y', w', y'', or w''.

Additionally, since the collection $\{X_j\}_{j=1}^d$ is involutive it follows that R_0 and R'' vanish if $y_0 = z_0 = y'' = z'' = 0$, and hence every nonvanishing term in the Taylor expansions of R_0 and R'' must contain a factor other than (y', z'), similarly for $\tilde{R}_0$ and $\tilde{R}''$. Putting this together with the fact that $R(x,y,z) = 0$ if $z = y$, we can write

$$R_j(x,y,z) = \sum_{\substack{|\alpha|+|\beta|=2\\|\beta|\ge 1}} c_{j,\alpha,\beta}(x)\, y^\alpha (z-y)^\beta + \sum_{\substack{|\alpha|+|\beta|=3\\|\beta|\ge 1}} c_{j,\alpha,\beta}(x,y,z)\, y^\alpha (z-y)^\beta, \tag{3.5}$$

for smooth functions $c_{j,\alpha,\beta}$, where $c_{j,\alpha,\beta}\equiv 0$ unless $\operatorname{order}(y^\alpha z^\beta)<\operatorname{order}(y_j)$, and also unless one of α', β', α'', or β'' is nonzero. Additionally, if $j=0$ or $j\geq d+1$ then $c_{j,\alpha,\beta}\equiv 0$ unless one of α_0, β_0, α'', or β'' is nonzero.

The same conditions also hold on $\tilde c_{j,\alpha,\beta}$ in the following expansion of $\tilde R(x,y,w)$,

$$\tilde R_j(x,y,w)=\sum_{\substack{|\alpha|+|\beta|=2\\ |\beta|\geq 1}}\tilde c_{j,\alpha,\beta}(x)\,y^\alpha w^\beta+\sum_{\substack{|\alpha|+|\beta|=3\\ |\beta|\geq 1}}\tilde c_{j,\alpha,\beta}(x,y,z)\,y^\alpha w^\beta.$$

4. The semiclassical calculus on U

Recall that

$$m(\eta)=\Big(|\eta_0|^6+|\eta'|^{12}+|\eta''|^4\Big)^{\frac{1}{12}}\approx|\eta_0|^{\frac12}+|\eta'|+|\eta''|^{\frac13},$$

which is smooth for $\eta\neq 0$ and homogeneous of degree 1, in that $m(\delta_r(\eta))=rm(\eta)$.

We assume given a compact subset $K\Subset U$, and choose r_1 so that the exponential map $y\to\exp_x(y)$ is a diffeomorphism on the ball $\{|y|\leq r_1\}$ for all $x\in K$, and fix $r_0<r_1$ such that

$$\bigcup_{\tilde x\in\exp_x(B_{r_0})}\exp_{\tilde x}(B_{r_0})\subset\exp_x(B_{r_1}).$$

We fix functions $\chi_j\in C_{\mathrm c}^\infty(B_{r_j})$ with $\chi_0(y)=1$ for $|y|\leq\frac12 r_0$ and

$$\chi_1(\Theta_x(\cdot))=1\ \text{ on a neighborhood of }\ \bigcup_{\tilde x}\operatorname{supp}\Big(\chi_0(\Theta_x(\tilde x))\,\chi_0(\Theta_{\tilde x}(\cdot))\Big).$$

Given a symbol $a(x,\eta)\in C^\infty(U\times\mathbb R^{2d+1})$ supported where $x\in K$, we define

$$a_h(x,\eta)=a(x,\eta_0,\eta',h\eta''),$$

and define a nonisotropic semiclassical quantization of a by the rule

$$\begin{aligned}a_h(x,hD)f(x)&=\frac{1}{(2\pi h)^{2d+1}}\int_{\mathbb R^{4d+2}}e^{-i\langle y,\eta\rangle/h}\,a_h(x,\eta)\,\chi_0(y)\,f(\exp_x(y))\,dy\,d\eta\\&=\frac{1}{(2\pi)^{2d+1}}\int_{\mathbb R^{2d+1}}e^{-i\langle y,\eta\rangle}\,a_h(x,\eta)\,\chi_0(hy)\,f(\exp_x(hy))\,dy\,d\eta.\end{aligned}\tag{4.1}$$

Thus the Schwartz kernel of $a_h(x,hD)$ is supported in $K\times K_{r_0}$, where K_{r_0} is the image of $K\times\overline B_{r_0}$ under $(x,y)\to\exp_x(y)$.

If $p(x,\eta)=\sum_{|\alpha|\leq n}c_\alpha(x)\,\eta^\alpha$ is a polynomial in η, then

$$\big(p_h(x,hD)f\big)(x)=p_h(x,(-i\partial_y))\,f(\exp_x(hy))\big|_{y=0}.$$

In particular, we have the following correspondence of symbols to operators:

$$i\eta_j\ :\ hX_j,\quad 0\leq j\leq d,\qquad i\eta_j\ :\ h^2X_j,\quad d+1\leq j\leq 2d.\tag{4.2}$$

Suppose that the symbol a satisfies homogeneous order-0 type estimates of the form

$$|\partial_x^\beta\partial_\eta^\alpha a(x,\eta)|\leq C_{\alpha,\beta}\,m(\eta)^{-\operatorname{order}(\alpha)}.$$

The uncertainty principal, used for example for proving L^2 continuity of $a_h(x, hD)$, requires uniform bounds on $\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x,\eta)$. On the other hand,

$$\begin{aligned}|\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x,\eta)| &= h^{|\alpha_0|+|\alpha'|+2|\alpha''|} |(\partial_x^\alpha \partial_\eta^\alpha a)_h(x,\eta)| \\ &\le C_\alpha\, h^{|\alpha_0|+|\alpha'|+2|\alpha''|}\, m_h(\eta)^{-2|\alpha_0|-|\alpha'|-3|\alpha''|}.\end{aligned}$$

To have uniform bounds as $h \to 0$ would require truncating $a(x,\eta)$ to where $m(\eta) \ge h^{\frac{1}{2}}$. It is convenient to work with bounded symbols, hence for symbols of order n we will multiply by a factor of $h^{-\frac{n}{2}}$ to make symbols of any order be of size $\lesssim 1$ when $m(\eta) = h^{\frac{1}{2}}$.

Definition 4.1. Let $m(h,\eta) = (h^{\frac{1}{2}} + m(\eta))$. A h-dependent family of symbols $a(x,\eta)$ belongs to $S^n(m)$ if, for all α, β, there is $C_{\alpha,\beta}$ independent of h such that, for $0 < h \le 1$,

$$|\partial_x^\beta \partial_\eta^\alpha a(x,\eta)| \le C_{\alpha,\beta}\, h^{-\frac{n}{2}} m(h,\eta)^{n-\mathrm{order}(\alpha)}.$$

We use $S_h^n(m)$ to denote symbols of the form $a_h(x,\eta)$ with $a(x,\eta) \in S^n(m)$, and $\Psi_h^n(m)$ to denote operators $a_h(x,hD)$ arising from symbols $a \in S^n(m)$.

We will show that the symbol of an operator is uniquely determined. For polynomial symbols we note the following result,

$$h^{-\frac{1}{2}\,\mathrm{order}(\alpha)} \eta^\alpha \in S^{\mathrm{order}(\alpha)}(m).$$

By (4.2) we then have the following examples.

$$\begin{aligned}&X_0 \in \Psi_h^2(m),\\ &h^{\frac{1}{2}} X_j \in \Psi_h^1(m), \quad 1 \le j \le d,\\ &h^{\frac{1}{2}} X_j \in \Psi_h^3(m), \quad d+1 \le j \le 2d.\end{aligned}$$

A more general example of a symbol in $S^n(m)$ is $h^{-\frac{n}{2}} a(\eta)\big(1-\phi(h^{-\frac{1}{2}} m(\eta))\big)$ where $a(\delta_r \eta) = r^n a(\eta)$.

It is easy to verify the following properties:

$$\begin{aligned}&S^n(m) \cdot S^{n'}(m) \subset S^{n+n'}(m).\\ &S^n(m) \supset S^{n'}(m) \quad \text{if} \quad n' < n.\\ &a \in S^n(m) \quad \Rightarrow \quad h^{\frac{1}{2}\,\mathrm{order}(\alpha)} \partial_\eta^\alpha \partial_x^\beta a \in S^{n-\mathrm{order}(\alpha)}.\end{aligned}$$

Definition 4.2. Given a sequence of symbols $a_j \in S^{n-j}(m)$ we say that $a \sim \sum_j a_j$ if for all N

$$a - \sum_{j=0}^{N-1} a_j \in S^{n-N}(m).$$

Consequently, a is uniquely determined up to a symbol in $S^{-\infty}(m) = \bigcap_N S^{-N}(m)$.

We note the following simple result as an example of a $S^{-\infty}(m)$.

(4.3) If $\phi \in \mathcal{S}(\mathbb{R})$ and $\phi(s) = 1$ when $|s| \le 1$ then $\phi(h^{-\frac{1}{2}} m(\eta)) \in S^{-\infty}(m)$.

Lemma 4.1. *Suppose that $a_j \in S^{n-j}(m)$, $j \in \mathbb{N}$. Then there exists $a \in S^n(m)$ with $a \sim \sum_j a_j$.*

Proof. We will construct a sequence of positive real numbers $R_j \to \infty$ such that for all N,

$$\sum_{j=N}^{\infty} \Big(1 - \phi(R_j^{-1} h^{-\frac12} m(\eta))\Big) a_j(x,\eta) \;\text{ converges in } S^{n-N}(m). \tag{4.4}$$

Defining a to be this sum where $N=0$ gives the result, we see by (4.3) that

$$\phi(R_j^{-1} h^{-\frac12} m(\eta)) \in S^{-N}(m) \;\text{ for all } N.$$

Additionally, the $S^0(m)$ seminorms of $\phi(R_j^{-1} h^{-\frac12} m(\eta))$ are uniformly bounded independent of R. The result (4.4) follows if we choose R_j so that for all $|\alpha| + |\beta| \le j$,

$$\Big|\partial_x^\beta \partial_\eta^\alpha \Big(1 - \phi(R_j^{-1} h^{-\frac12} m(\eta))\Big) a_j(x,\eta)\Big| \le 2^{-j} h^{-\frac{n+1-j}{2}} m(h,\eta)^{n+1-j-\mathrm{order}(\alpha)}.$$

Such R_j can be chosen by observing that on the support of $1 - \phi(R_j^{-1} h^{-\frac12} m(\eta))$ we have

$$h^{\frac12} m(h,\eta)^{-1} \le (1+R_j)^{-1}.$$

$\square$

Let ϕ and ψ generate a smooth Littlewood-Paley decomposition of $[0,\infty)$:

$$1 = \phi(s) + \sum_{j=1}^{\infty} \psi(2^{-j}s)\,, \quad \mathrm{supp}(\phi) \subset [0,2), \;\; \mathrm{supp}(\psi) \subset (\tfrac12, 2).$$

Given a symbol $a \in S^n(m)$, we make the following decomposition,

$$\begin{aligned} a(x,\eta) &= \phi(h^{-\frac12} m(\eta)) a(x,\eta) + \sum_{j=1}^{\infty} \psi(h^{-\frac12} 2^{-j} m(\eta)) a(x,\eta) \\ &= \sum_{j=0}^{\infty} a_j(x,\eta). \end{aligned} \tag{4.5}$$

Then a_j is supported where $(h^{\frac12} + m(\eta)) \approx 2^j h^{\frac12}$, and thus

$$\big|\partial_x^\beta \partial_\eta^\alpha a_j(x,\eta)\big| \le C_{\alpha,\beta}\, 2^{jn} \big(2^j h^{\frac12}\big)^{-\,\mathrm{order}(\alpha)}.$$

It follows that $a_0(x, \delta_{h^{1/2}}(\eta)) \in C_c^\infty(K \times \{|\eta| < 8\})$ with bounds uniform over h, and for $j \ge 1$ that $2^{-nj} a_j(x, \delta_{2^j h^{1/2}}(\eta))$ is uniformly bounded in $C_c^\infty(K \times \{\frac18 < |\eta| < 8\})$ over h and j.

Theorem 4.2. *Suppose that $a \in S^n(m)$ and $b \in S^{n'}(m)$. Then there is $c \in S^{n+n'}(m)$ so that*

$$a_h(x,hD) \circ b_h(x,hD) = c_h(x,hD) + R_h,$$

where R_h is a smoothing operator with kernel $R_h(x,\tilde x)$ supported in $K \times K_{r_0}$ satisfying, for all N,

$$|\partial_x^\alpha \partial_{\tilde x}^\beta R_h(x,\tilde x)| \le C_{N,\alpha,\beta}\, h^N.$$

Furthermore, $c - a\sharp b \in S^{n+n'-1}(m)$, where $(a\sharp b)(x,\xi) = \big(a(x,\cdot)\sharp b(x,\cdot)\big)(\xi)$.

Proof. We start with the first part of the result. For $x \in K$ and $h > 0$ we can write

$$\chi_1(\Theta_x(\tilde{x}))f(\tilde{x}) = \frac{1}{(2\pi h)^{2d+1}} \int e^{i\langle \Theta_x(\tilde{x}),\xi\rangle/h - i\langle y,\xi\rangle/h}\, \chi_1(y)\, f(\exp_x(y))\, dy\, d\xi.$$

Since $a_h(x,hD)b_h(x,hD)f(x) = a_h(x,hD)b_h(x,hD)\Big(\chi_1(\Theta_x(\cdot))f\Big)(x)$, we can write

$$a_h(x,hD)\circ b_h(x,hD)f(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y,\xi\rangle/h}\, c_h(x,\xi)\, \chi_1(y)\, f(\exp_x(y))\, dy\, d\xi$$

where

$$c_h(x,\xi) = \Big(a_h(x,hD)\, b_h(x,hD) e^{i\langle \Theta_x(\cdot),\xi\rangle/h}\Big)(x).$$

We will show that $c_h(x,\eta) \in S_h^{n+n'}(m)$. To estimate the term R_h coming from replacing χ_1 by χ_0 we write $\chi_1(y) - \chi_0(y) = |y|^{2N}\beta_N(y)$, where $\beta_N \in C_c^\infty$. Then $R_h f(x) = \int K(x, \exp_x(y)) f(y)\, dy$ where

$$K(x,\exp_x(y)) = \frac{\beta_N(y)}{(2\pi h)^{2d+1}} \int e^{-i\langle y,\xi\rangle/h}\, (h^2\Delta_\eta)^N c_h(x,\xi)\, d\xi.$$

It is simple to verify that $(h^2\Delta_\eta)^N c_h(x,\xi) \in S_h^{n+n'-2N}(m)$, thus R_h corresponds to the restriction to $2|y| \geq r_0$ of an integral kernel with $S^{-\infty}(m)$ symbol. The kernel satisfies (5.3), and the result then follows after changing variables $y = \Theta_x(\tilde{x})$.

Let a_i and b_j be the nonisotropic Littlewood-Paley decomposition of a and b as in (4.5), and define c_{ij} by

$$(c_{ij})_h(x,\xi) = \Big((a_i)_h(x,hD)\, (b_j)_h(x,hD) e^{i\langle \Theta_x(\cdot),\xi\rangle/h}\Big)(x),$$

so that $c = \sum_{ij} c_{ij}$. From (4.1) we can write $(c_{ij})_h(x,\xi)$ as

$$\frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y,\eta\rangle - i\langle w,\zeta\rangle + ih^{-1}\langle \tilde{\Theta}(x,hy,hw),\xi\rangle}\, (a_i)_h(x,\eta)\, (b_j)_h(\exp_x(hy),\zeta)$$
$$\times\, \chi_0(hy)\, \chi_0(hw)\, dw\, d\zeta\, dy\, d\eta.$$

Consider first the case that $i \geq j$. Substitute the quantity $h^{-1}\Theta(x,hy,hz)$ for w to write this as

$$\frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y,\eta\rangle - ih^{-1}\langle \Theta(x,hy,hz),\zeta\rangle + i\langle z,\xi\rangle}\, a_i(x,\eta_0,\eta',h\eta'')\, b_j(\exp_x(hy),\zeta_0,\zeta',h\zeta'')$$
$$\times\, \chi_0(hy)\, \chi_0(\Theta(x,hy,hz))\, |D_z\Theta|(x,hy,hz)\, dz\, d\zeta\, dy\, d\eta.$$

By the comments following (4.5), we write

$$b_j(\exp_x(hy),\zeta_0,\zeta',h\zeta'')\, \chi_0(hy)\, \chi_0(\Theta(x,hy,hz))\, |D_z\Theta|(x,hy,hz)$$
$$= 2^{jn'}\tilde{b}_j(x,hy,hz,(2^{-2j}h^{-1}\zeta_0, 2^{-j}h^{-\frac{1}{2}}\zeta', 2^{-3j}h^{-\frac{1}{2}}\zeta''))$$

where $\tilde{b}_j \in C_c^\infty(K \times B_{r_0} \times B_{r_1} \times B_8)$, with bounds uniform over h and j, and a similar representation holds for a_i with 2^j replaced by 2^i. We make a nonisotropic dilation of ζ and η by the factors $(2^{-2j}h^{-1}, 2^{-j}h^{-\frac{1}{2}}, 2^{-3j}h^{-\frac{1}{2}})$, and of z and y by the reciprocal factors, to write

$$c_{ij}(x,\xi) = 2^{j(n+n')}\tilde{c}_{ij}(x, \delta_{2^{-j}h^{-1/2}}(\xi)),$$

where $\tilde c_{ij}(x,\xi)$ is given by

$$(4.6)\quad \frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle - i\langle\bar\Theta_y(z)+R(h,x,y,z),\zeta\rangle + i\langle z,\xi\rangle}\,\tilde a_i(x,4^{j-i}\eta_0,2^{j-i}\eta',8^{j-i}\eta'')$$
$$\times\,\tilde b_j(x,2^{-2j}y_0,2^{-j}h^{\frac12}y',2^{-3j}h^{\frac12}y'',2^{-2j}z_0,2^{-j}h^{\frac12}z',2^{-3j}h^{\frac12}z'',\zeta)\,dz\,d\zeta\,dy\,d\eta,$$

with

$$\begin{aligned}\langle R(h,x,y,z),\zeta\rangle &= 2^{2j}\,R_0(x,2^{-2j}y_0,2^{-j}h^{\frac12}y',2^{-3j}h^{\frac12}y'',2^{-2j}z_0,2^{-j}h^{\frac12}z',2^{-3j}h^{\frac12}z'')\,\zeta_0\\ &\quad+2^jh^{-\frac12}R'(x,2^{-2j}y_0,2^{-j}h^{\frac12}y',2^{-3j}h^{\frac12}y'',2^{-2j}z_0,2^{-j}h^{\frac12}z',2^{-3j}h^{\frac12}z'')\cdot\zeta'\\ &\quad+2^{3j}h^{-\frac12}R''(x,2^{-2j}y_0,2^{-j}h^{\frac12}y',2^{-3j}h^{\frac12}y'',2^{-2j}z_0,2^{-j}h^{\frac12}z',2^{-3j}h^{\frac12}z'')\cdot\zeta''.\end{aligned}$$

By the support condition on $\tilde b$ we have $|\zeta|\le 8$. Also, if $i\ge1$ then $\tilde a(x,\eta)=0$ when $|\eta|\le\frac18$.

We next apply the expansion (3.5). The condition on order$(y^\alpha z^\beta)$ insures that we bring out strictly more powers of 2^{-j} than needed to cancel the powers of 2^j in front, and since there is at least one factor of (y',z',y'',z'') we also bring out a factor $h^{\frac12}$ to cancel off the $h^{-\frac12}$ in front. We conclude that, on the support of the integrand,

$$|R(h,x,y,z)|\le C\,2^{-j}|z-y|\,\Big(\,|y|+|z-y|+|y|^2+|z-y|^2\Big),$$

and also

$$(4.7)\qquad |\partial_x^\gamma\partial_y^\alpha\partial_z^\beta R(h,x,y,z)|\le C_{\gamma,\alpha,\beta}\,2^{-j}\Big(1+|y|^3+|z-y|^3\Big).$$

Additionally, if we let $w=\bar\Theta(x,y,z)+R(h,x,y,z)$, then with analogous notation we see from (3.1) and Lemma 3.2 that $z=\bar\Theta_{-y}(w)+\tilde R(h,x,y,w)$, where

$$|\tilde R(h,x,y,w)|\le C\,2^{-j}|w|\,\Big(\,|y|+|w|+|y|^2+|w|^2\Big).$$

Consequently, since $\tilde\Theta$ is the inverse function to Θ for fixed y, uniformly over j we have

$$\big|\bar\Theta_y(z)+R(h,x,y,z)\big|\le C\,|z-y|\,\Big(1+|y|^2+|z-y|^2\Big),$$
$$|z-y|\le C\,\big|\bar\Theta_y(z)+R(h,x,y,z)\big|\,\Big(1+|y|^2+\big|\bar\Theta_y(z)+R(h,x,y,z)\big|^2\Big)$$

and hence

$$(1+|y|^2)^{-1}|z-y|\le C\,\big|\bar\Theta_y(z)+R(h,x,y,z)\big|\,\Big(1+\big|\bar\Theta_y(z)+R(h,x,y,z)\big|^2\Big).$$

Considering the function

$$g_{ij}(x,y)=\frac{1}{(2\pi)^{4d+2}}\int e^{-i\langle y,\eta\rangle}\,\tilde a_i(x,4^{j-i}\eta_0,2^{j-i}\eta',8^{j-i}\eta'')\,d\eta,$$

simple estimates show that

$$(4.8)\qquad |\partial_x^\alpha\partial_y^\beta g_{ij}(x,y)|\le C_{N,\alpha,\beta}2^{(4d+2)(i-j)}\big(1+2^{2(i-j)}|y_0|+2^{i-j}|y'|+2^{3(i-j)}|y''|\big)^{-N}.$$

Additionally, if $i>j$, hence $i\ge1$, then $\tilde a_i(x,\eta)$ vanishes for $|\eta|\le\frac18$, hence can be assumed to be of the form $|\eta|^2\tilde a_i(x,\eta)$ for similar $\tilde a_i(x,\eta)$, and so we can write

$$g_{ij}(x,y)=\sum_{|\gamma|=2}2^{(j-i)\,\mathrm{order}(\gamma)}\partial_y^\gamma g_{ij,\gamma}(x,y),$$

where $g_{ij,\gamma}(x,y)$ satisfies the same estimates (4.8) as $g_{ij}(x,y)$. Similarly, if we let $f_j(x,y,z)$ equal

$$\int e^{-i\langle\bar\Theta_y(z)+R(h,x,y,z),\zeta\rangle}\,\tilde b_j(x,2^{-2j}y_0,2^{-j}h^{\frac12}y',2^{-3j}h^{\frac12}y'',2^{-2j}z_0,2^{-j}h^{\frac12}z',2^{-3j}h^{\frac12}z'',\zeta)\,d\zeta,$$

then

$$|\partial_x^\alpha\partial_y^\beta\partial_z^\theta f_j(x,y,z)|\le C_{N,\alpha,\beta,\theta}\Big(1+|y|+|y-z|\Big)^{3(|\alpha|+|\beta|+|\theta|)}\Big(1+(1+|y|^2)^{-1}|y-z|\Big)^{-N}. \tag{4.9}$$

This leads to the bound, for all N,α,β,

$$\left|\partial_x^\alpha\partial_z^\beta\int g_{ij}(x,y)\,f_j(x,y,z)\,dy\right|\le C_{N,\alpha,\beta}\,2^{-2(j-i)}(1+|z|)^{-N}, \tag{4.10}$$

which leads to uniform (over i and j) Schwartz bounds on $2^{2(i-j)}\tilde c_{ij}(x,\xi)$.

In the case $i\ge 1$, hence $j\ge 1$ as well, we can harmlessly divide $\tilde b_j$ by $|\zeta|^{2\ell}$. Since

$$\Big((\partial_{z_0}-y'\cdot\partial_{z''},\partial_{z'}+y_0\partial_{z''},\partial_{z''})-\zeta\Big)e^{-i\langle\bar\Theta_y(z),\zeta\rangle}\;=\;0,$$

and ∂_z acting on any other factor of z in the integrand for f_j leads to a gain of 2^{-j}, we can then write, for each $\ell\in\mathbb N$ and $k\in\mathbb N$,

$$\tilde c_{ij}(x,\xi)=\sum_{|\gamma|\le 2l}2^{-2(i-j)}2^{-j(2\ell-|\gamma|)}\xi^\gamma\tilde c_{ij,\ell,\gamma}(x,\xi), \tag{4.11}$$

for Schwartz functions $c_{ij,\ell,\gamma}$ which are uniformly bounded over i,j.

In case $j\ge i$, we can similarly write $c_{ij}(x,\xi)=2^{i(n+n')}\tilde c_{ij}(x,\delta_{2^{-i}h^{-1/2}}(\xi))$, where

$$\tilde c_{ij}(x,\xi)=\sum_{|\gamma|\le 2l}2^{-2(j-i)}2^{-i(2\ell-|\gamma|)}\xi^\gamma\tilde c_{ij,\ell,\gamma}(x,\xi),$$

for Schwartz functions $c_{ij,\ell,\gamma}$ which are uniformly bounded over i,j. The analysis is similar to the case $i\ge j$, using instead the following representation for $c_{ij}(x,\xi)$,

$$\frac{1}{(2\pi)^{4d+2}}\int e^{-ih^{-1}\langle\tilde\Theta(x,hv,-hw),\eta\rangle-i\langle w,\zeta\rangle+i\langle v,\xi\rangle}\,a_i(x,\eta_0,\eta',h\eta'')\,b_j(\exp_{\exp_x(hv)}(-hw),\zeta_0,\zeta',h\zeta'')$$
$$\times\,\chi_0(\tilde\Theta(x,hv,-hw))\,\chi_0(hw)\,|D_v\tilde\Theta|(x,hv,-hw)\,dw\,d\zeta\,dv\,d\eta.$$

It thus suffices to show that $\sum_{i\ge j}2^{j(n+n')}\tilde c_{ij}(x,\delta_{2^{-j}h^{-1/2}}(\xi))\in S^{n+n'}(m)$. We prove that

$$\Big|\sum_{i\ge j}2^{j(n+n')}\tilde c_{ij}(x,\delta_{2^{-j}h^{-1/2}}(\xi))\Big|\le C\left(1+h^{-\frac12}m(\xi)\right)^{n+n'},$$

and estimates on derivatives will follow similarly. We use (4.11) to bound the sum by

$$C_{N,\ell}\sum_{i\ge j\ge 0}\;\sum_{|\gamma|\le 2\ell}2^{j(n+n')}2^{-2(i-j)}2^{-j(2\ell-|\gamma|)}(2^{-j}h^{-\frac12}m(\xi))^{\mathrm{order}(\gamma)}\big(1+2^{-j}h^{-\frac12}m(\xi)\big)^{-N}.$$

Fix $2\ell>n+n'$. If $h^{-\frac12}m(\xi)\le 1$ the sum is easily bounded by a constant. If $h^{-\frac12}m(\xi)\ge 1$, we bound this by

$$C_{N,\ell}\sum_{j=0}^\infty 2^{j(n+n')}(2^{-j}h^{-\frac12}m(\xi))^{2\ell}\big(1+2^{-j}h^{-\frac12}m(\xi)\big)^{-N},$$

where we use that order$(\gamma) \geq |\gamma|$. This leads to the desired bound by separating into the cases $2^j \geq h^{-\frac{1}{2}} m(\xi)$ and $2^j \leq h^{-\frac{1}{2}} m(\xi)$.

We now turn to the proof that $c - a\sharp b \in S^{n+n'-1}$. The proof shows that there is a full asymptotic expansion for c in terms of functions of x and h times the $\sharp$ composition of derivatives of a and b, but the leading term is the important result.

To start, we note that by (4.8), the integral in (4.10) over the region $|y| \geq 2^{j\epsilon}$, for any fixed $\epsilon > 0$, leads to a gain of 2^{-jN} in the form (4.11) for $\tilde{c}_{ij}$. Similarly, if $|y| \leq 2^{j\epsilon}$, then (4.9) shows that the integral over the region $|z| \geq 2^{3j\epsilon}$ has a similar gain.

Thus, up to changing c by a term in $S^{-\infty}(m)$, we can insert a factor $\phi(2^{-j\epsilon}y)\phi(2^{-3j\epsilon}z)$ in the formula (4.6) defining $\tilde{c}_{ij}$. We now fix $\epsilon = \frac{1}{9}$. By (4.7), on the support of this cutoff we have

$$|\partial_x^\gamma \partial_y^\alpha \partial_z^\beta R(h,x,y,z)| \leq C_{\gamma,\alpha,\beta}. \tag{4.12}$$

We next take a Taylor expansion in (y,z) of the $\tilde{b}_j$ term in (4.6) about $y=z=0$, to write it as

$$\begin{aligned} &\sum_{|\alpha|+|\beta|\leq N} h^{\frac{1}{2}(|\alpha'+\beta'|+|\alpha''+\beta''|)} 2^{-j\,\mathrm{order}(\alpha+\beta)} y^\alpha z^\beta \partial_y^\alpha \partial_z^\beta \tilde{b}_j(x,0,0,\zeta) \\ &\quad + \sum_{|\alpha|+|\beta|=N} h^{\frac{1}{2}(N-\alpha_0-\beta_0)} 2^{-j\,\mathrm{order}(\alpha+\beta)} y^\alpha z^\beta \\ &\qquad\qquad \times b_{j,\alpha,\beta}(x, 2^{-2j}y_0, 2^{-j}h^{\frac{1}{2}}y', 2^{-3j}h^{\frac{1}{2}}y'', 2^{-2j}z_0, 2^{-j}h^{\frac{1}{2}}z', 2^{-3j}h^{\frac{1}{2}}z'', \zeta) \end{aligned}$$

where we recall that $\tilde{b}_j \in C_c^\infty(K \times B_{r_0} \times B_{r_1} \times B_8)$, with bounds uniform over h and j. The term $\alpha = \beta = 0$ is just $2^{-jn} b_j(x,\zeta)$, since $|D_z\Theta(x,0,0)| = 1$. The α, β term will lead to a symbol of order $n + n' - \mathrm{order}(\alpha+\beta)$, so up to terms in $S^{n+n'-1}(m)$ we have that $(2\pi)^{4d+2}\tilde{c}_{ij}$ is given by the integral

$$\int e^{-i\langle y,\eta\rangle - i\langle \bar{\Theta}_y(z) + R(h,x,y,z),\zeta\rangle + i\langle z,\xi\rangle} \tilde{a}_i(x, 4^{j-i}\eta_0, 2^{j-i}\eta', 8^{j-i}\eta'') \tilde{b}_j(x,\zeta) \phi(2^{-j\epsilon}y)\phi(2^{-3j\epsilon}z)\, dz\, d\zeta\, dy\, d\eta.$$

This expression with $R \equiv 0$ leads, modulo a term in $S^{-\infty}(m)$, to the symbol $\big(a(x,\cdot)\sharp b(x,\cdot)\big)(\xi)$. Thus we need show the difference leads to a symbol in $S^{n+n'-1}$. To estimate the difference, we expand

$$e^{-i\langle R(h,x,y,z),\zeta\rangle} = \sum_{k=0}^N \frac{(-i)^k \langle R(h,x,y,z),\zeta\rangle^k}{k!} \;+\; \langle R(h,x,y,z),\zeta\rangle^N\, e_N\big(\langle R(h,x,y,z),\zeta\rangle\big)$$

with $e_N\big(\langle R(h,x,y,z),\zeta\rangle\big)$ having bounded derivatives of all order on the support of the integrand by (4.12), recalling that $|\zeta| \leq 1$ on the support of the integrand. Inserting e_N into (4.10) preserves that estimate, so it suffices to observe that, if we set

$$\begin{aligned} \tilde{c}_{ij,k}(x,\xi) = \int e^{-i\langle y,\eta\rangle - i\langle \bar{\Theta}_y(z),\zeta\rangle + i\langle z,\xi\rangle} \tilde{a}_i(x, 4^{j-i}\eta_0, 2^{j-i}\eta', 8^{j-i}\eta'') \tilde{b}_j(x,\zeta) \\ \times \langle R(h,x,y,z),\zeta\rangle^k \phi(2^{-j\epsilon}y)\phi(2^{-3j\epsilon}z)\, dz\, d\zeta\, dy\, d\eta \end{aligned}$$

then we have uniform Schwartz bounds on $2^{jk}\, 2^{|i-j|}\tilde{c}_{ij,k}(x,\xi)$, which follows from the above proof and (4.7). $\square$

Corollary 4.3. *Suppose P_h is as in* (1.1). *Given $\rho \in C_c^\infty(K^o)$, there is a symbol $q(x,\xi) \in S^{-2}(m)$, with principal symbol $h\rho(x)(1-\chi(h^{-\frac12}m(\xi)))q(\xi)$, so that*

$$q_h(x,hD)\circ P_h = \rho(x) \mod \Psi^{-\infty}(m).$$

Proof. Fix $\tilde\rho(x) \in C_c^\infty(K)$ with $\tilde\rho = 1$ on a neighborhood of $\operatorname{supp}(\rho)$. Define

$$\tilde q(\xi) = h(1-\chi(h^{-\frac12}m(\xi)))q(\xi)$$

where $q(\xi)$ is the Fourier transform of the fundamental solution for $Y_0 - \sum_{j=1}^d Y_j^2$. We observe that $\tilde\rho(x)\tilde q(\xi) \in S^{-2}(m)$, and that

$$\tilde\rho(x)q_h(hD)\circ P_h = \tilde\rho(x) - a_h(x,hD), \qquad a \in S^{-1}(m).$$

To see this, note that $h\sum_{j=1}^d c_j(x)X_j \in S^1(m)$, the symbol of $h^{\frac12}X_j$ is $i\eta_j$ for $1\le j\le d$, and the symbol of X_0 is η_0. We use Theorem 4.2 and Lemma 2.3 to see that the left hand side equals, modulo an operator in $\Psi_h^{-1}(m)$, the operator with symbol

$$\tilde\rho(x)(1-\chi((h^{-\frac12}m(\xi)))\Big(i\tilde q\sharp\eta_0 - \sum_{j=1}^d \tilde q\sharp\eta_j\sharp\eta_j\Big) = \tilde\rho(x)(1-\chi((h^{-\frac12}m(\xi))) = \tilde\rho(x) \mod S^{-\infty}(m).$$

Next let $b_h(x,hD) \in \Psi_h^0(m)$ be an asymptotic sum $b_h(x,hD) \sim I + \sum_{n=1}^\infty a_h(x,hD)^n$. Then

$$\rho(x)b_h(x,hD)\circ\tilde\rho(x)q_h(hD)\circ P_h = \rho(x)b_h(x,hD)\circ\Big(\tilde\rho(x) - a_h(x,hD)\Big).$$

By pseudo-locality of $b_h(x,hD)$ this differs by a term in $\Psi_h^{-\infty}(m)$ from

$$\rho(x)b_h(x,hD)\circ\Big(I - a_h(x,hD)\Big) = \rho(x) \mod \Psi_h^{-\infty}(m).$$

The principal symbol of $q_h(x,hD) = \rho(x)b_h(x,hD)\circ\tilde\rho(x)\tilde q_h(hD)$ is then $\rho(x)\tilde q(\xi)$. $\square$

5. L^2 boundedness for order 0 operators

Let ϕ and ψ generate a smooth Littlewood-Paley decomposition of $[0,\infty)$:

$$1 = \phi(s) + \sum_{j=1}^\infty \psi(2^{-j}s)\,, \quad \operatorname{supp}(\phi)\subset[0,2], \ \ \operatorname{supp}(\psi)\subset[\tfrac12,2].$$

Given a symbol $a \in S^n(m)$, we make the following decomposition,

$$\begin{aligned} a(h,x,\eta) &= \phi(h^{-\frac12}m(\eta))a(h,x,\eta) + \sum_{j=1}^\infty \psi(h^{-\frac12}2^{-j}m(\eta))a(h,x,\eta) \\ &= \sum_{j=0}^\infty a_j(h,x,\eta). \end{aligned}$$

Then a_j is supported where $(h^{\frac12}+m(\eta)) \approx 2^jh^{\frac12}$, and

$$\big|\partial_x^\beta\partial_\eta^\alpha a_j(x,\eta)\big| \le C_{\alpha,\beta}\,2^{jn}(2^jh^{\frac12})^{-\operatorname{order}(\alpha)}.$$

The operator $a_{j,h}(x,hD)$ is given by the following integral kernel on $U\times U$ with respect to the measure $dm(\tilde x)$, where $w(x,\tilde x)dm(\tilde x)=\exp_x^*(dy)$,

$$K_j(x,\tilde x)=w(x,\tilde x)\chi_0(\Theta_x(\tilde x))\int e^{-i\langle\Theta_x(\tilde x),\eta\rangle}a_{j,h}(x,h\eta)\,d\eta.$$

The symbol $a_{j,h}(x,h\eta)$ is supported in the rectangle $|\eta_0|\le 2^{2j}$, $|\eta'|\le 2^jh^{-\frac12}$, $|\eta''|\le 2^{3j}h^{-\frac12}$, with derivative estimates in each variable inversely proportional to the respective sidelengths. Consequently, there are Schwartz functions $\rho_j(x,y)$, supported for $x\in\operatorname{supp}(a)$ with Schwartz norms independent of j, so that

$$(w^{-1}K_j)(x,\exp_x(y))=2^{jn}\,2^{j(2+4d)}h^{-d}\rho_j\Big(x,2^{2j}y_0,2^jh^{-\frac12}y',2^{3j}h^{-\frac12}y''\Big)\chi_0(y),\tag{5.1}$$

and in particular, for all N

$$|K_j(x,\tilde x)|\le 2^{jn}\,2^{j(2+4d)}h^{-d}\,C_N\Big(1+2^{2j}|\Theta_x(\tilde x)_0|+2^jh^{-\frac12}|\Theta_x(\tilde x)'|+2^{3j}h^{-\frac12}|\Theta_x(\tilde x)''|\Big)^{-N}.\tag{5.2}$$

If $a\in S^{-\infty}$ then (5.1) holds for all $n\in\mathbb Z$, and we obtain the following.

Corollary 5.1. *If $a\in S^{-\infty}(m)$, then $a_h(x,hD)$ is given by a smooth integral kernel $K_h(x,\tilde x)$ in the measure $dm(\tilde x)$, so that for some Schwartz function $\rho(x,y)$, supported for $x\in\operatorname{supp}(a)$,*

$$(w^{-1}K)(x,\exp_x(y))=h^{-d}\rho(x,y_0,h^{-\frac12}y',h^{-\frac12}y'')\chi_0(y).\tag{5.3}$$

We next observe that the vector fields $2^{-2j}Y_0$, $2^{-j}h^{\frac12}Y'$, and $2^{-3j}h^{\frac12}Y''$ acting as differential operators in y all preserve the form of $w^{-1}K_j$; that is, they give an expression of the same form with ρ_j uniformly bounded over j in each Schwartz seminorm.

The same holds for the operators $2^{-2j}X_0$, $2^{-j}h^{\frac12}X'$, and $2^{-3j}h^{\frac12}X''$, acting on $K_j(x,\tilde x)$ as differential operators in either the x or $\tilde x$ variables. For action in the $\tilde x$ variable this follows by Lemma 3.1, where we use that there is at least one factor of y' or y'' in the expansion of $R_0(x,y,\partial_y)$ to compensate for the factor of $h^{-\frac12}$ coming from $\partial_{y'}$ and $\partial_{y''}$ terms in the expansion of X_0. For action in the x variable we work in coordinates $x=\exp_{\tilde x}(y)$, hence $\tilde x=\exp_x(-y)$, to write

$$(w^{-1}K_j)(\exp_{\tilde x}(y),\tilde x)=2^{jn}\,2^{j(2+4d)}h^{-d}\rho_j\Big(\exp_{\tilde x}(y),-2^{2j}y_0,-2^jh^{-\frac12}y',-2^{3j}h^{-\frac12}y''\Big)\chi_0(-y).\tag{5.4}$$

To summarize, for $a\in S^n(m)$, we can write

$$\begin{aligned}(2^{-2j}X_0)^{\alpha_0}&(2^{-j}h^{\frac12}X')^{\alpha'}(2^{-3j}h^{\frac12}X'')^{\alpha''}K_j(x,\tilde x)\\&=2^{jn}\,2^{j(2+4d)}h^{-d}\rho_{j,\alpha}\Big(x,2^{2j}\Theta_x(\tilde x)_0,2^jh^{-\frac12}\Theta_x(\tilde x)',2^{3j}h^{-\frac12}\Theta_x(\tilde x)''\Big)\chi_\alpha(x,\tilde x)\end{aligned}\tag{5.5}$$

where the functions $\rho_{j,\alpha}$ and χ_α satisfy seminorm bounds that depend on α, but are uniform over j and h. This holds whether any given vector X in the product acts on x or $\tilde x$.

Conversely, suppose that $j\ge1$, so that $a_j(x,\delta_{2^{-j}h^{-1/2}}(\eta))\in C_c^\infty(K\times\{\frac18\le|\eta|\le 8\})$. Then, for any ℓ, dividing $a_{j,h}$ by the appropriate nonisotropic dilation of $|\eta|^{2\ell}$ and taking the Fourier transform shows that we can write

$$\begin{aligned}(w^{-1}K_j)(x,\exp_x(y))&=2^{jn}\,2^{j(2+4d)}h^{-d}\times\\&\sum_{|\alpha|=2\ell}\chi_\alpha(x,y)(2^{-2j}\partial_{y_0})^{\alpha_0}(2^{-j}h^{\frac12}\partial_{y'})^{\alpha'}(2^{-3j}h^{\frac12}\partial_{y''})^{\alpha''}\rho_{j,\alpha}\Big(x,2^{2j}y_0,2^jh^{-\frac12}y',2^{3j}h^{-\frac12}y''\Big),\end{aligned}$$

for Schwartz functions $\rho_{j,k,\alpha,\beta}$ that are uniformly bounded over j, and $\chi_\alpha \in C_{\mathrm{c}}^\infty(K \times B_{r_0})$.

Using Lemma 3.1, we write

$$\begin{aligned}
\partial_{y_0} &= X_0 + y' \cdot X'' - R_0(x,y,\partial_y) - y' \cdot R''(x,y,\partial_y),\\
\partial_{y'} &= X' - y_0\, X'' - R'(x,y,\partial_y) + y_0\, R''(x,y,\partial_y),\\
\partial_{y''} &= X'' - R''(x,y,\partial_y).
\end{aligned}$$

where the X_j act in y. Substituting this into $R(x,y,\partial_y)$, and using that the X_j form a smooth frame, we can expand each ∂_{y_j} as a finite sum over $2 \le |\alpha| \le 3$,

$$\begin{aligned}
\partial_{y_0} &= X_0 + y' \cdot X'' + \sum_{\alpha,k} c_{0,\alpha,k}(x,y)\, y^\alpha X_k, && && \operatorname{order}(y_k) - \operatorname{order}(\alpha) < 2,\\
\partial_{y_j} &= X_j - y_0\, X_{j+d} + \sum_{\alpha,k} c_{j,\alpha,k}(x,y)\, y^\alpha X_k, && 1 \le j \le d, && \operatorname{order}(y_k) - \operatorname{order}(\alpha) < 1,\\
\partial_{y_j} &= X_j + \sum_{\alpha,k} c_{j,\alpha,k}(x,y)\, y^\alpha X_k, && d+1 \le j \le 2d, && \operatorname{order}(y_k) - \operatorname{order}(\alpha) < 3.
\end{aligned}$$

Additionally, $c_{0,\alpha,k} \equiv 0$ unless either $\alpha' \neq 0$ or $\alpha'' \neq 0$.

Letting $\bar X_j$ denote the transpose of the differential operator X_j with respect to dy, it follows that, with the X_j acting on y, we can write

$$\begin{aligned}
(w^{-1}K_j)(x, \exp_x(y)) &= 2^{jn}\, 2^{j(2+4d)} h^{-d} \times\\
&\sum_{|\alpha|=2\ell} \chi_\alpha(x,y) \big(2^{-2j} \bar X_0\big)^{\alpha_0} \big(2^{-j} h^{\frac12} \bar X'\big)^{\alpha'} \big(2^{-3j} h^{\frac12} \bar X''\big)^{\alpha''} \rho_{j,\alpha}\Big(x, 2^{2j} y_0, 2^j h^{-\frac12} y', 2^{3j} h^{-\frac12} y''\Big),
\end{aligned}$$

where the $\rho_{j,\alpha}$ may depend on h, but with uniform Schwartz bounds over $0 \le h \le 1$ and $j \in \mathbb{N}$. Expressing the action of $\bar X$ in terms of $\tilde x$, this leads to the following expansion

$$K_j(x,\tilde x) = \sum_{|\alpha|=2\ell} \sum_{\beta \le \alpha} 2^{-j \operatorname{order}(\alpha)} \big(\bar X_0\big)^{\beta_0} \big(h^{\frac12} \bar X'\big)^{\beta'} \big(h^{\frac12} \bar X''\big)^{\beta''} K_{j,\alpha,\beta}(x,\tilde x), \tag{5.6}$$

for kernels $K_{j,\alpha,\beta}$ satisfying (5.2) with C_N depending on ℓ but uniform over j, α, β. Here we can take $\bar X$ to be the transpose with respect to $dm(\tilde x)$, since that differs from the transpose with respect to dy by a smooth function.

Theorem 5.2. *If $a \in S^0(m)$, then $a_h(x, hD)$ is a bounded linear operator on $L^2(U)$, with operator norm depending only on a finite number of seminorm bounds for a. In particular, the operator norm is uniformly bounded over $0 < h \le 1$.*

Proof. We decompose $a_h(x,hD) = \sum_{j=0}^\infty a_{j,h}(x,hD)$. Using (5.1) and (5.4) it is easily verified that the kernel $K_j(x,\tilde x)$ of $a_{j,h}(x,hD)$ satisfies the Schur test,

$$\sup_x \int K_j(x,\tilde x)\, dm(\tilde x) \le C\,, \qquad \sup_{\tilde x} \int K_j(x,\tilde x)\, dm(x) \le C.$$

We deduce L^2 boundedness from the Cotlar-Stein lemma (see [KS71] or [Ste93]), by showing that, for any $N \in \mathbb{N}$,

$$\|a_{i,h}(x,hD)^* a_{j,h}(x,hD)\|_{L^2 \to L^2} + \|a_{i,h}(x,hD) a_{j,h}(x,hD)^*\|_{L^2 \to L^2} \le C\, 2^{-N|i-j|}, \tag{5.7}$$

for C uniform over h and j. If $i=j$ this follows from L^2 boundedness of each term, so without loss of generality we consider $j>i\geq 0$, and in particular $j\geq 1$. Given $\ell\in\mathbb{N}$ we then write the integral kernel of $a_{i,h}(x,hD)a_{j,h}(x,hD)^*$ as

$$\begin{aligned}\int K_i(x,w)\overline{K_j(\tilde{x},w)}\,dm(w)&\\ &=\int K_i(x,w)\sum_{|\alpha|=2\ell}\sum_{\beta\leq\alpha}2^{-j\,\mathrm{order}(\alpha)}\big(\bar{X}_0\big)^{\beta_0}\big(h^{\frac12}\bar{X}'\big)^{\beta'}\big(h^{\frac12}\bar{X}''\big)^{\beta''}\overline{K_{j,\alpha,\beta}(x,w)}\,dm(w)\\ &=\sum_{|\alpha|=2\ell}\sum_{\beta\leq\alpha}2^{i\,\mathrm{order}(\beta)-j\,\mathrm{order}(\alpha)}\int K_{i,\beta}(x,w)\,\overline{K_{j,\alpha,\beta}(x,w)}\,dm(w)\end{aligned}$$

where $K_{i,\beta}(x,w)=\big(2^{-2i}X_0\big)^{\beta_0}\big(2^{-i}h^{\frac12}X'\big)^{\beta'}\big(2^{-3i}h^{\frac12}X''\big)^{\beta''}K_i(x,w)$, and in all cases X acts on w. Since $i\,\mathrm{order}(\beta)-j\,\mathrm{order}(\alpha)<2\ell(i-j)$, using (5.5) and the Schur test on the composition, as well as symmetry of the estimates in x and $\tilde{x}$, we obtain (5.7) with $N=2\ell$. $\square$

We note the following result for $a\in S^n(m)$, which holds since $2^{-jn}a_j(x,\eta)\in S^0(m)$,

$$\sup_{j\geq 0}2^{-jn}\|a_j(x,hD)f\|_{L^2(U)}\leq C\,\|f\|_{L^2(U)},\qquad a\in S^n(m). \tag{5.8}$$

6. Estimates on $S^*(M)$

Let (M,g) be a compact Riemannian manifold of dimension $d+1$, and $S^*(M)\subset T^*(M)$ its unit cosphere bundle. We consider the Hamiltonian function $\frac12|\zeta|^2_{\mathrm{g}(z)}=\sum_{i,j=1}^{d+1}\mathrm{g}^{ij}(z)\zeta_i\zeta_j$, and recall that $S^*(M)$ is the level set $|\zeta|_{\mathrm{g}(z)}=1$. We use $X_0=H$ to denote the Hamiltonian field for $\frac12|\zeta|^2_{\mathrm{g}}$,

$$X_0=\sum_{j=1}^{d+1}\mathrm{g}^{ij}(z)\zeta_i\partial_{z_j}-\frac12\sum_{k=1}^{d}\partial_{z_k}\mathrm{g}^{ij}(z)\zeta_i\zeta_j\partial_{\zeta_k},$$

which is tangent to $S^*(M)$.

We cover $S^*(M)$ by a finite collection of open coordinate charts as follows. Let $\{V_\alpha\}$ form a finite covering of M by coordinate charts, over which we can identity $T^*(M)$ with $V_\alpha\times\mathbb{R}^{d+1}$ and $S^*(M)$ with $V_\alpha\times\mathbb{S}^d$. We cover $\mathbb{S}^d$ by two coordinate charts $W^\pm$ over each of which there is a section of the frame bundle. We thus obtain a cover of $S^*(M)$ by open sets $\{V_\alpha\times W^\pm\}$, which by counting each V_α twice we can label as U_α, such that on U_α there is an orthonormal collection $\{X_j\}_{j=1}^d$ of vector fields that are tangent to $S^*_z(V_\alpha)$ for each $z\in V_\alpha$. The X_j then form an orthonormal frame on the tangent space of $S^*_z(V_\alpha)$. The collection $\{X_j\}_{j=1}^d$ is involutive, since they span the tangent space of the leaves of a foliation.

There is a natural isometric identification $T_\zeta(T^*_x(M))\sim T_z(M)$, which identifies $\{X_j\}_{z,\zeta}$ with an orthonormal collection of vectors $\{\tilde{X}_j\}_{z,\zeta}\in T_z(V_\alpha)$, which are also orthogonal to $(X_0)_{z,\zeta}$. One then verifies that

$$[X_j,X_0]=\sum_{j=1}^{d+1}\mathrm{g}^{ij}(z)X_j(\zeta_i)\partial_{z_j}\mod(T(S^*_z(M))=\tilde{X}_j\mod(T(S^*_z(M)).$$

Setting $X_{j+d}=-\frac12\tilde{X}_j$, we have $[X_0,X_j]-2X_{j+d}\in\mathrm{span}\{X_j\}_{j=1}^d$, so the assumptions of the introduction are satisfied on $\{X_j\}_{j=0}^{2d}$.

Let $\Delta_{\mathbb{S}}$ be the induced non-negative Laplacian acting on the fibers $S^*_z(M)$ of the bundle, and let Δ be the non-negative Laplacian on $S^*(M)$. See for example [Dro17, Section 2.1] for details, where it is shown that Δ and $\Delta_{\mathbb{S}}$ commute. One verifies that, over each U_α, one has

$$\Delta_{\mathbb{S}} = -\sum_{j=1}^{d} X_j^2 + \sum_{j=1}^{d} c_j(z,\zeta) X_j.$$

We now use $x=(z,\zeta)\in\mathbb{R}^{2d+1}$ to denote the variables on $S^*(U_\alpha)$, and define

$$P_h = H + h\Delta_{\mathbb{S}} = X_0 - \sum_{j=1}^{d} hX_j^2 + \sum_{j=1}^{d} c_j(x) hX_j.$$

Thus on each U_α, the operator P_h differs from the sum of squares considered previously by an operator in $h^{\frac12}\Psi_h^1(m)$, and the pseudodifferential calculus shows that, given $\chi_\alpha\in C_{\mathrm{c}}^\infty(U_\alpha)$, there exists a symbol $a_\alpha\in S^{-2}(m)$, the quantization of which depends on χ_α through the choice of χ_0 in (4.1), so that on U_α we have

$$a_{\alpha,h}(x,hD)\circ P_h u = \chi_\alpha(x)u + r_{\alpha,h}(x,hD)u, \qquad r_h\in\Psi_h^{-\infty}(m).$$

Note that both $a_{\alpha,h}(x,hD)$ and $r_{\alpha,h}(x,hD)$ are properly supported in U_α. We now take a partition of unity χ_α subordinate to the cover U_α, and define

$$A_h v = \sum_\alpha a_{\alpha,h}(x,hD)v, \qquad R_h v = \sum_\alpha r_{\alpha,h}(x,hD)v.$$

Then $A_h\circ P_h = I + R_h$, and for all N_1, N_2 we have

$$\|(h\Delta)^{N_1} R_h\,(h\Delta)^{N_2} u\|_{L^2(S^*(M))} \le C_{N_1,N_2}\|u\|_{L^2(S^*(M))}. \tag{6.1}$$

This follows from Theorem 5.2 and the fact that $h\Delta\in\Psi_h^6(U_\alpha)$ for each α.

More generally, we define $\Psi_h^\sigma(m)$ on $S^*(M)$ as sums $\sum_\alpha a_{\alpha,h}(x,hD)$ with $a_\alpha\in S^\sigma(m)$ on U_α. The function χ_0 in the quantization (4.1) depends on the x support of $a_\alpha(x,\eta)$, which is always assumed to be a compact subset of U_α.

The semiclassical Sobolev spaces are defined on $S^*(M)$ using the spectral decomposition of Δ, with norm

$$\|f\|_{H_h^\sigma} = \|(1+h^2\Delta)^{\sigma/2} f\|_{L^2}.$$

We will consider cutoffs $\rho(s)$ satisfying, for some $c'>c>0$

$$\rho(s)\in C^\infty(\mathbb{R}), \qquad \rho(s)=0 \ \text{ if } s\le c, \quad \rho(s)=1 \ \text{ if } s\ge c'. \tag{6.2}$$

The operator $\rho(h^2\Delta)$ is then defined as a spectral multiplier. We observe the following simple result for $R_h\in\Psi_h^{-\infty}(m)$ on $S^*(M)$. For all N and σ we have

$$\|\rho(h^2\Delta)R_h(x,hD)u\|_{H_h^\sigma} + \|R_h(x,hD)\rho(h^2\Delta)u\|_{H_h^\sigma} \le C_{N,\sigma} h^N \|u\|_{L^2}. \tag{6.3}$$

This follows by writing $\rho(h^2\Delta)(1+h^2\Delta)^\sigma = f(h^2\Delta)\circ(h^2\Delta)^N$ for a bounded function $f(s)$, provided $N>\sigma$, and using (6.1).

Theorem 6.1. *Suppose that $\sigma\le 0$, that $A_h\in\Psi_h^\sigma(m)$, and ρ satisfies* (6.2). *Then*

$$\|\rho(h^2\Delta)a_h(x,hD)u\|_{H_h^{-\sigma/3}} + \|a_h(x,hD)\rho(h^2\Delta)u\|_{H_h^{-\sigma/3}} \le C\,h^{-\sigma/6}\|u\|_{L^2}.$$

Proof. Choose k so $6k+\sigma>0$. For each $h\in(0,1]$, we show that $a_h=a_{h,0}+a_{h,1}$ where

$$\|(h^2\Delta)^k a_{h,0}(x,hD)u\|_{L^2}+\|a_{h,0}(x,hD)(h^2\Delta)^k u\|_{L^2}+\|a_{h,1}(x,hD)u\|_{H_h^{\sigma/3}}\le C\,h^{-\sigma/6}\|u\|_{L^2}.$$

The result follows since $\rho(s)\le\min(s^k,1)$. Using the Littlewood-Paley decomposition as in the proof of Theorem 5.2, applied to each a_α in the sum defining a, we let

$$a_{h,0}(x,\eta)=\sum_{2^j\le h^{-\frac16}}a_{h,j}(x,\eta),\qquad\qquad a_{h,1}(x,\eta)=\sum_{2^j>h^{-\frac16}}a_{h,j}(x,\eta).$$

Recalling the form (5.5), we see that applying $h^2\Delta$ to $a_{h,j}(x,hD)$ is equivalent to multiplying it by at most $2^{6j}h$. As in the proof of (5.7) we conclude that

$$\|(1+h^2\Delta)^k a_{h,j}(x,hD)a_{h,i}(x,hD)^*(1+h^2\Delta)^k\|_{L^2\to L^2}\le(1+2^{6i}h)^k(1+2^{6j}h)^k\,2^{\sigma(i+j)-N|i-j|}\,.$$

For $2^j,2^i\ge h^{-\frac16}$, we interpolate with the L^2 bounds (5.8) to obtain

$$\|(1+h^2\Delta)^{-\sigma/6}a_{h,j}(x,hD)a_{h,i}(x,hD)^*(1+h^2\Delta)^{-\sigma/6}\|_{L^2\to L^2}\le C\,h^{-\sigma/3}\,2^{-N|i-j|}\,.$$

This estimate also holds for the transposed operators. The Cotlar-Stein lemma then implies the bounds for $a_{h,1}(x,hD)$.

Similarly, we have

$$\|(h^2\Delta)^k a_{h,j}(x,hD)\|_{L^2\to L^2}+\|a_{h,j}(x,hD)(h^2\Delta)^k\|_{L^2\to L^2}\le C\,(2^{6j}h)^k\,2^{\sigma j},$$

which we may sum over $2^j\le h^{-\frac16}$ to conclude the bounds involving $a_{h,0}(x,hD)$. □

Corollary 6.2. *Suppose that $\sigma\le0$, and $A_h\in\Psi_h^\sigma(m)$. Then*

$$\|(1+h\Delta)^{-\sigma/6}A_hu\|_{L^2}\le C\,\|u\|_{L^2}.$$

Proof. As in the proof of Theorem 6.1 we observe that, for $k=0,1,2,\ldots,$

$$\|(1+h\Delta)^k a_{h,j}(x,hD)a_{h,i}(x,hD)^*(1+h\Delta)^k\|_{L^2\to L^2}\le 2^{6k(i+j)}2^{\sigma(i+j)-N|i-j|}\,.$$

We interpolate between $k=0$ and $k>-6\sigma$ to obtain

$$\|(1+h\Delta)^{-\sigma/6}a_{h,j}(x,hD)a_{h,i}(x,hD)^*(1+h\Delta)^{-\sigma/6}\|_{L^2\to L^2}\le C\,2^{-N|i-j|}\,.$$

This estimate also holds for the transposed operators. The Cotlar-Stein lemma then implies the result. □

Theorem 6.3. *The following bound holds for $h\in(0,1]$, with R_h satisfying* (6.1).

$$\|Hu\|_{L^2}+h\|\Delta_{\mathbb S}u\|_{L^2}+\|(1+h\Delta)^{\frac13}u\|_{L^2}\le C\,\|P_hu\|_{L^2}+C_N\|(1+h\Delta)^{-N}u\|_{L^2}.$$

Proof. Write $u=A_hP_hu+R_hu$, and observe that HA_h, $h\Delta_{\mathbb S}A_h\in\Psi_h^0(m)$. We also observe that HR_h, $h\Delta_{\mathbb S}R_h\in\Psi_h^{-\infty}(m)$, hence satisfy the same bounds (6.1) as R_h. Since $h\Delta$ commutes with $(1+h\Delta)^{\frac13}$, we see that $(1+h\Delta)^{\frac13}R_h$ also satisfies the bounds (6.1). The result then follows by Corollary 6.2. □

Theorem 6.4. *Suppose that ρ_1 and ρ_2 satisfy* (6.2)*, and $\rho_2 = 1$ on a neighborhood of* $\operatorname{supp}(\rho_1)$*. Given $R > 0$, the following holds for all N, and all $|\lambda| \le R$ and $h \in (0,1]$,*

$$h^{-\frac{1}{3}}\|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{\frac{1}{3}}\sum_{j=1}^{d}\|X_j\rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2}$$
$$\le C_{N,R}\Big(\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|u\|_{L^2}\Big).$$

Proof. We follow the scheme of the proof of Theorem 2 of [Dro17], using the parametrix A_h of P_h to replace the positive commutator arguments. Write

$$\rho_1(h^2\Delta)u = A_h\rho_1(h^2\Delta)(P_h-\lambda)u + A_h[P_h,\rho_1(h^2\Delta)]u + \lambda A_h\rho_1(h^2\Delta)u + R_h\rho_1(h^2\Delta)u.$$

To handle the commutator term on the right, we use that $[\Delta_{\mathbb{S}},\rho_1(h^2\Delta)] = 0$, so $[P_h,\rho_1(h^2\Delta)] = [X_0,\rho_1(h^2\Delta)]$. Now let $\tilde\rho_1(s)$ be any function satisfying (6.2) which equals 1 on a neighborhood of $\operatorname{supp}(\rho_1)$. Then following [Dro17], we use that the essential support of $[X_0,\rho_1(h^2\Delta)]$ is contained within the elliptic set of $\tilde\rho(h^2\Delta)$, and we can thus bound

$$\|[P_h,\rho_1(h^2\Delta)]u\|_{L^2} \le C\,\|\tilde\rho_1(h^2\Delta)]u\|_{L^2} + C_N h^N\|u\|_{L^2}.$$

Applying Theorem 6.1 and (6.3) we obtain

$$h^{-\frac{1}{3}}\|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{-\frac{1}{6}}\sum_{j=1}^{d}\|h^{\frac{1}{2}}X_j\rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2}$$
$$\le C\Big(\|\rho_1(h^2\Delta)(P_h-\lambda)u\|_{L^2} + \|\tilde\rho_1(h^2\Delta)u\|_{L^2} + |\lambda|\|\rho_1(h^2\Delta)u\|_{L^2}\Big) + C_N\,h^N\|u\|_{L^2}.$$

For h bounded away from 0 we can absorb the term $|\lambda|\|\rho_1(h^2\Delta)u\|_{L^2}$ into $C_N\,h^N\|u\|_{L^2}$, and for h small we can subtract it from both sides.

From this we deduce the following bound, for any such $\tilde\rho_1$,

$$\|\rho_1(h^2\Delta)u\|_{L^2} \le C_{N,R}\Big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde\rho_1(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\Big).$$

We now choose a sequence of cutoffs $\tilde\rho_j$ for $1 \le j \le 3N$, satisfying (6.2), such that for all j we have $\tilde\rho_{j+1} = 1$ on a neighborhood of $\operatorname{supp}(\tilde\rho_j)$, and $\rho_2 = 1$ on a neighborhood of $\operatorname{supp}(\tilde\rho_j)$. Then replacing ρ_1 by $\tilde\rho_j$, the preceding estimate shows that

$$\|\tilde\rho_j(h^2\Delta)u\|_{L^2} \le C_{N,R}\Big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde\rho_{j+1}(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\Big).$$

We conclude by iteration that

$$\|\tilde\rho_1(h^2\Delta)u\|_{L^2} \le C_{N,R}\Big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|\rho_2(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}\Big)$$
$$\le C_{N,R}\Big(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|u\|_{L^2}\Big).$$

Together with the above this yields the statement of the theorem. □

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