UNIFORM RESOLVENT ESTIMATES ON MANIFOLDS OF BOUNDED CURVATURE

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ABSTRACT. We establish $L^{q*} \to L^q$ bounds for the resolvent of the Laplacian on compact Riemannian manifolds assuming only that the sectional curvatures of the manifold are uniformly bounded. When the resolvent parameter lies outside a parabolic neighborhood of $[0, \infty)$, the operator norm of the resolvent is shown to depend only on upper bounds for the sectional curvature and diameter and lower bounds for the volume. The resolvent bounds are derived from square-function estimates for the wave equation, an approach that admits the use of paradifferential approximations in the parametrix construction.

1. INTRODUCTION

In this paper we assume that (M, g) is a compact Riemannian manifold of dimension d with uniformly bounded sectional curvatures. More precisely, we assume that for constants $K, D < \infty$, and $V > 0$, the sectional curvatures, diameter, and volume satisfy

$$|\sec(M)| \le K, \qquad \operatorname{diam}(M) \le D, \qquad \operatorname{vol}(M) \ge V. \tag{1.1}$$

It was shown by Cheeger [6] that under these conditions there is a lower bound $i(d, K, D, V)$ on the injectivity radius. The purpose of this paper is to show that certain $L^p \to L^q$ mapping bounds hold for the resolvent of the Laplacian on (M, g), with constants that depend only on K, D, V. As a corollary of the proof we show that the constants in the Strichartz estimates, which were established on such manifolds by Chen-Smith [7], also depend only on K, D, V. Concerning resolvent bounds, we prove the following.

Theorem 1.1. *Suppose that (M, g) satisfies (1.1), and $d \ge 3$. Given $\delta > 0$, there is a constant $C_0 = C(\delta, K, D, V, d)$ such that for all $z \in \mathbb{C}$ with $\operatorname{Im}(z) \ge \delta$ the following holds*

$$\|(-\Delta_{\mathrm{g}} - z^2)^{-1} f\|_{L^q(M)} \le C_0 \|f\|_{L^{q*}(M)},$$

provided that

$$\frac{1}{q*} - \frac{1}{q} = \frac{2}{d}, \qquad q \ge \frac{2(d+1)}{d-1}, \quad \textit{and} \quad q* \le \frac{2(d+1)}{d+3}.$$

For $q = 2n/(n-2)$, uniform resolvent bounds on smooth Riemannian manifolds were proven in Dos Santos Ferreira-Kenig-Salo [8] using the Hadamard parametrix. The dependence of C_0 on (M, g) was not explicitly considered, but controlling the Hadamard parametrix ruses a number of derivatives of g that increases with the dimension. The work of Bourgain-Shao-Sogge-Yao [4] developed the subject further, utilizing the connection of resolvent estimates and the spectral projection bounds of Sogge [14]. In addition, they

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showed that the domain of z over which uniform estimates hold cannot be enlarged in case (M, g) is the round sphere, but can be improved logarithmically for manifolds of nonpositive curvature, and by a power for the flat torus. In comparison, Euclidean space uniform resolvent bounds for $q = 2n/(n-2)$ hold by Kenig-Ruiz-Sogge [9] over the entire complex plane.

Uniform estimates for the same range of indices as in Theorem 1.1 were established on smooth Riemannian manifolds by Shao-Yao [12], using Sogge's spectral projection bounds to handle certain nonlocal terms. As with the works cited above, the proof relied on the Hadamard parametrix for local terms, and methods of Carleson-Sjölin concerning $L^p \to L^q$ bounds for oscillatory integral operators. More recently, Burq-Dos Santos Ferreira-Krupchyk [5] generalized the estimates to certain non-selfadjoint perturbations of the Laplacian. Their proof used dispersive properties of the resolvent in spatial directions, similar to the method used in Mockenhaupt-Seeger-Sogge [11] to establish squarefunction bounds in L^q for solutions to the Cauchy problem for the wave equation.

Along other lines, Chen-Smith [7] established both Strichartz estimates and squarefunction bounds for the Cauchy problem for the wave equation on Riemannian manifolds of bounded sectional curvature. This work generalized earlier work of Smith [13] and Tataru [15] establishing similar estimates for metrics with bounded second-order derivatives. An important step in these works is reducing matters to establishing dispersive estimates for a paradifferential approximation to the Laplacian. This method introduces a remainder term that is bounded in L^2 and can be handled as a driving force for the wave equation. This method can only work, however, for estimates that factor through L^2, and does not apply directly to other types of $L^p \to L^q$ mapping bounds for the exact solution. A key step in proving Theorem 1.1 is to first show that resolvent estimates can be derived directly from an inhomogeneous version of the squarefunction estimates of [11]. We state these estimates as Theorem 2.1 below, and prove them with constant depending only on the allowed quantities. The bounds on the constant are obtained using the methods of [7] in local coordinates, combined with results of Anderson and Cheeger [6], [2] and [3] concerning existence of a suitable cover by harmonic coordinate charts on (M, g).

As a corollary of these steps together with the local Strichartz estimates of [7], we will also obtain that the full range of inhomogeneous Strichartz estimates for the wave equation hold on (M, g), and that the dependence of the bounds on the geometry of g arises only from K, D, V. Recall that a triple (s, q, r) with $2 \leq q, r \leq \infty$ is said to be *admissible* for the wave equation if

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s, \qquad \frac{1}{q} \leq \frac{d-1}{2}\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

We then have the following.

Theorem 1.2. *Suppose that (M, g) and q are as in Theorem 1.1, and (s, q, r) and $(1-s, \tilde{q}, \tilde{r})$ are admissible, with $r, \tilde{r} < \infty$. Then for all $T > 0$ there exists $C = C(T, K, D, V, d)$, such that solutions to the Cauchy problem* (2.1) *satisfy*

$$\|u\|_{L^q([-T,T];L^r(M))} + \|u\|_{L^\infty([-T,T];H^s(M))} + \|\partial_t u\|_{L^\infty([-T,T];H^{s-1}(M))}$$
$$\leq C\Big(\|f\|_{H^s(M)} + \|g\|_{H^{s-1}(M)} + \|F\|_{L^{\tilde{q}'}([-T,T];L^{\tilde{r}'}(M))} \Big).$$

2. RESOLVENT ESTIMATES FROM SQUAREFUNCTION ESTIMATES

Consider the wave equation on a compact Riemannian manifold (M,g),

$$\begin{aligned}(\partial_t^2-\Delta_{\mathrm{g}})u(t,x)&=F(t,x),\\ u(0,x)&=f(x),\\ \partial_t u(0,x)&=g(x).\end{aligned}\tag{2.1}$$

In this section we derive Theorem 1.1 directly from the following square-function estimates for solutions to (2.1). These square-function estimates, which are an extension of the results of [7] for the homogeneous equation, will be derived in Section 3.

Theorem 2.1. *Suppose that* (M,g) *and* q *are as in Theorem 1.1. Then for all* $T>0$ *there exists* $C_1=C(T,K,D,V,d)$*, such that solutions to* (2.1) *satisfy*

$$\|u\|_{L^q(M;L^2([0,T]))}\le C_1\Big(\|f\|_{H^{s(q)}(M)}+\|g\|_{H^{s(q)-1}(M)}+\|F\|_{L^{q*}(M;L^2([0,T]))}\Big),$$

where $s(q)=d\Big(\frac12-\frac1q\Big)-\frac12$.

Observe that for q in the range of Theorem 1.1 we have

$$\frac{d-1}{2(d+1)}\le s(q)\le\frac{d+3}{2(d+1)},$$

and in particular $s(q)<1$. The Sobolev norm $H^s(M)$ used in Theorem 2.1 is the one defined intrinsically on M using the spectral decomposition of $-\Delta_{\mathrm{g}}$, though as we will show in Section 3 this norm is equivalent to one naturally defined using a suitable covering of M by local coordinate charts, provided that $-2\le s\le 2$.

An immediate consequence of Theorem 2.1 is the following. Let Π_n denote projection onto the span of eigenfunctions with eigenvalue $-\lambda_j^2$ such that $n\le\lambda_j<n+1$. Then for a constant C_1 as in Theorem 2.1,

$$\Big\|\Big(\sum_{n=0}^{2\langle\lambda\rangle}|\Pi_n f|^2\Big)^{1/2}\Big\|_{L^q(M)}\le C_1\,\langle\lambda\rangle^{s(q)}\|f\|_{L^2(M)}.\tag{2.2}$$

To see this, let $Qf=\sum_{n=0}^\infty n\Pi_n f$. Then for all q we have

$$\Big\|\Big(\sum_{n=0}^{\infty}|\Pi_n f|^2\Big)^{1/2}\Big\|_{L^q(M)}=\|\exp(itQ)f\|_{L^q(M;L^2([0,2\pi]))}.$$

From the fact that

$$(\partial_t-i\sqrt{-\Delta_{\mathrm{g}}})\exp(itQ)f=i(Q-\sqrt{-\Delta_{\mathrm{g}}})\exp(itQ)f$$

we obtain

$$\exp(itQ)f=\exp(it\sqrt{-\Delta_{\mathrm{g}}})f+i\int_0^t\exp(i(t-s)\sqrt{-\Delta_{\mathrm{g}}})(Q-\sqrt{-\Delta_{\mathrm{g}}})\exp(isQ)f\,ds.$$

Let $f_s=\exp(-is\sqrt{-\Delta_{\mathrm{g}}})(Q-\sqrt{-\Delta_{\mathrm{g}}})\exp(isQ)f$, and observe that $\|f_s\|_{L^2(M)}\le\|f\|_{L^2(M)}$. If f is spectrally localized to eigenvalues with $\lambda_j\le 2\langle\lambda\rangle+1$, then

$$\begin{aligned}\|\exp(itQ)f\|&_{L^q(M;L^2([0,2\pi]))}\\&\le\|\exp(it\sqrt{-\Delta_{\mathrm{g}}})f\|_{L^q(M;L^2([0,2\pi]))}+\int_0^{2\pi}\|\exp(it\sqrt{-\Delta_{\mathrm{g}}})f_s\|_{L^q(M;L^2([0,2\pi]))}\\&\le(2\pi+1)C_1\langle\lambda\rangle^{s(q)}\|f\|_{L^2(M)}.\end{aligned}$$

Corollary 2.2. *Suppose that $z=\lambda+i\mu$ with $\mu\ge\delta>0$, and that the spectrum of f is contained in $0\le\lambda_j<2\langle\lambda\rangle$. Then the following hold under the conditions of Theorem 2.1, where $C_1=C(\delta,K,D,V,d)$ is independent of λ,*

$$\begin{aligned}\|(-\Delta_g-z^2)^{-1}f\|_{L^q(M)}&\le C_1\langle\lambda\rangle^{s(q)}|z|^{-1}\mu^{-\frac12}\|f\|_{L^2(M)},\\\|(-\Delta_g-z^2)^{-1}f\|_{L^2(M)}&\le C_1\langle\lambda\rangle^{s(q)}|z|^{-1}\mu^{-\frac12}\|f\|_{L^{q'}(M)},\\\|(-\Delta_g-z^2)^{-1}f\|_{L^2(M)}&\le 2|z|^{-1}\mu^{-1}\|f\|_{L^2(M)}.\end{aligned}$$

Proof. The last bound follows since $|\lambda_j^2-z^2|\ge\max(2|\lambda\mu|,|\mu^2-\lambda^2|)\ge\frac12|z|\mu$. By duality it then suffices to prove the first inequality. We write

$$\begin{aligned}(-\Delta_{\mathrm{g}}-z^2)^{-1}f&=\sum_{n=0}^{2\langle\lambda\rangle}(-\Delta_{\mathrm{g}}-z^2)^{-1}\Pi_nf\\&=\sum_{n=0}^{2\langle\lambda\rangle}(n^2-z^2)^{-1}\Pi_n(Sf),\end{aligned}$$

where Sf is the spectral multiplier given by

$$Sf=\sum_{n=0}^{\infty}(n^2-z^2)\big(-\Delta_{\mathrm{g}}-z^2\big)^{-1}\Pi_nf.$$

If $\mu\ge\delta$ and $\lambda_j\in[n,n+1)$, then

$$|n^2-z^2|\le C(\delta)\,|\lambda_j^2-z^2|,$$

and consequently $\|Sf\|_{L^2(M)}\le C(\delta)\|f\|_{L^2(M)}$. We now bound

$$\begin{aligned}\Big|\sum_{n=0}^{2\langle\lambda\rangle}(n^2-z^2)^{-1}\Pi_n(Sf)\Big|&\le\Big(\sum_{n=0}^{\infty}\frac{1}{|n^2-z^2|^2}\Big)^{1/2}\Big(\sum_{n=0}^{2\langle\lambda\rangle}|\Pi_n(Sf)|^2\Big)^{1/2}\\&\le\frac{C\delta^{-1/2}}{|z|\mu^{1/2}}\Big(\sum_{n=0}^{2\langle\lambda\rangle}|\Pi_n(Sf)|^2\Big)^{1/2}.\end{aligned}$$

The result now follows by (2.2). □

Proof of Theorem 1.1. Let $z=\lambda+i\mu$ where $\mu\ge\delta$. By (3.2), we can decompose f using spectral cutoffs into two parts, spectrally localized respectively to $\lambda_j<2\langle\lambda\rangle$ and $\lambda_j>\frac32\langle\lambda\rangle$, in a manner that is continuous on $L^q(M)$ for all $1<q<\infty$.

Suppose first that spec(f) is contained in $0 \le \lambda_j < 2\langle\lambda\rangle$. Let $u(t,\cdot) = e^{-itz} f$, so that

$$(\partial_t^2 - \Delta_{\mathrm{g}})u = e^{-itz}(-\Delta_{\mathrm{g}} - z^2)f, \qquad u(0,\cdot) = f, \qquad \partial_t u(0,\cdot) = -izf(x).$$

Since $\langle\lambda\rangle \le C(\delta)|z|$, we can apply Corollary 2.2 with $q = q{*}'$ to conclude

$$\begin{aligned}\|f\|_{H^{s(q)}(M)} + |z|\|f\|_{H^{s(q)-1}(M)} &\le C(\delta)\langle\lambda\rangle^{s(q)-1}|z|\|f\|_{L^2(M)} \\ &\le C_1\lambda^{s(q)+s(q{*}')-1}\mu^{-1/2}\|(-\Delta_{\mathrm{g}} - z^2)f\|_{L^{q*}(M)} \\ &= C_1\mu^{-1/2}\|(-\Delta_{\mathrm{g}} - z^2)f\|_{L^{q*}(M)},\end{aligned}$$

where we use that $s(q) + s(q{*}') = 1$. Also, we have

$$\|e^{-itz}(-\Delta_{\mathrm{g}} - z^2)f\|_{L^{q*}(M;L^2([0,1]))} = (e^{2\mu} - 1)^{1/2}\mu^{-1/2}\|(-\Delta_{\mathrm{g}} - z^2)f\|_{L^{q*}(M)},$$

and

$$\|e^{-itz}f\|_{L^q(M;L^2([0,1]))} = (e^{2\mu} - 1)^{1/2}\mu^{-1/2}\|f\|_{L^q(M)}.$$

Since $e^{2\mu} - 1 \ge e^{2\delta} - 1 > 0$, applying Theorem 2.1 yields

$$\|f\|_{L^q(M)} \le C_1\|(-\Delta_{\mathrm{g}} - z^2)f\|_{L^{q*}(M)}.$$

Next suppose that spec(f) is contained in $\lambda_j > \frac{3}{2}\langle\lambda\rangle$. Note that $(1 + \lambda_j^2)/(\lambda_j^2 - z^2)$ is uniformly bounded if $\mathrm{Re}(z) = \lambda$ and $\lambda_j \ge \frac{3}{2}\langle\lambda\rangle$, hence $(1 - \Delta_{\mathrm{g}})/(-\Delta_{\mathrm{g}} - z^2)$ is uniformly bounded on $L^2(M)$. We then have

$$\begin{aligned}\|f\|_{L^q(M)} &\le C\,\|(1 - \Delta_{\mathrm{g}})^{\frac{d}{2}(\frac{1}{2} - \frac{1}{q})} f\|_{L^2(M)} \\ &\le C\,\|(1 - \Delta_{\mathrm{g}})^{-\frac{d}{2}(\frac{1}{q*} - \frac{1}{2})}(-\Delta_{\mathrm{g}} - z^2)f\|_{L^2(M)} \\ &\le C\,\|(-\Delta_{\mathrm{g}} - z^2)f\|_{L^{q*}(M)},\end{aligned}$$

where we use the gap relation $\frac{1}{q*} - \frac{1}{q} = \frac{2}{d}$ and the Sobolev embedding bound (3.4). ☐

3. INHOMOGENEOUS SQUAREFUNCTION ESTIMATES

In this section we derive the inhomogeneous squarefunction estimates of Theorem 2.1 from the results of Chen-Smith [7], taking care to show that the constant C_0 depends only on the relevant geometric quantities. Throughout this section, C will denote a constant that depends only on the constants K, D, V of (1.1) in addition to the dimension d, and $C(a)$ a constant that depends only on K, D, V, d, a.

We start with the result of Cheeger and Anderson [6], [2] and [3], that shows that under the condition (1.1), for each $c_0 > 0$ and $d < p < \infty$ there exists $r_h > 0$, and a collection of N harmonic coordinate charts $F_\nu : U_\nu \to B(r_h) = \{x : |x| < r_h\}$, so that $F_\nu^{-1}(B(r_h/2))$ cover M, and there are uniform bounds on the metric tensor g_{ij} in the local coordinate charts

$$\sup_{x\in B(r_h)} |\mathrm{g}_{ij}(x) - \delta_{ij}| \le c_0, \qquad \|\partial g_{ij}\|_{L^p(B(r_h))} \le c_0. \tag{3.1}$$

Furthermore, the numbers N and r_h depend only on the quantities K, D, V, d, p, c_0.

In addition, there is such a harmonic coordinate chart on any ball in M of diameter r_h. This yields that $\mathrm{Vol}(B(x,r)) \sim r^d$ for $r < r_h/2$, hence a global doubling condition

$$\mathrm{vol}(B(x,2r)) \le C\,\mathrm{vol}(B(x,r)).$$

A result of Li and Yau [10, Theorem 3.2] then yields small time exponential bounds on the heat kernel $H(t,x,y)$,

$$0 \le H(t,x,y) \le C\,t^{-\frac{d}{2}}\,\exp\Bigl(\frac{-\rho(x,y)^2}{5t}\Bigr), \qquad 0 < t \le 1,$$

where $\rho(x,y)$ is the geodesic distance on (M,g). By Alexopoulos [1, Theorem 6.1], together with the doubling condition this implies a version of the Hörmander-Mikhlin multiplier theorem. In particular, it yields that if $\phi \in C_c^\infty(\mathbb{R})$, then

$$\|\phi((-\Delta_{\mathrm{g}})^{1/2}/\langle\lambda\rangle)\|_{L^q(M)} \le C(\phi,q)\|f\|_{L^q(M)}, \tag{3.2}$$

uniformly over $\lambda \in \mathbb{R}$.

The bounds on g_{ij} and $|\sec(M)|$ yield uniform upper bounds on the coefficients R_{ijkl} of the Riemann curvature tensor in the local harmonic coordinates. Using cutoffs and elliptic regularity as in the proof of Lemma 2.1 in [7], given a coordinate chart $F_\nu : U_\nu \to B(r_h)$, we may find metrics g_{ij} on $\mathbb{R}^d$ which agree with the induced metric for $|x| \le .9r_h$, and such that $\mathrm{g}_{ij} = \delta_{ij}$ for $|x| \ge r_h$, and

$$\|\partial_x^2 \mathrm{g}_{ij}\|_{BMO} + \|\mathrm{g}_{ij}\|_{\mathrm{Lip}} + \|\mathrm{R}_{ijkl}\|_{L^\infty} \le C, \quad \|\mathrm{g}_{ij} - \delta_{ij}\|_{L^\infty} \le c_0. \tag{3.3}$$

It remains to note that we can take a partition of unity on M of the form $\chi_\nu \circ F_\nu$, with $\mathrm{supp}(\chi_\nu) \subset B(.6r_h)$, and with $\|\chi_\nu\|_{W^{2,p}} \le C(p)$ for all $p < \infty$. To see this, we observe that by (3.1) the Riemannian distance function on M satisfies, for $x, y \in B(.9r_h)$,

$$.1|x-y| \le d_M(F_\nu^{-1}(x), F_\nu^{-1}(y)) \le 1.1|x-y|,$$

provided that c_0 is sufficiently small. In this case $F_\nu(U_\nu \cap U_\mu)$ contains the $r_h/100$ neighborhood of $B(.6r_h) \cap F_\nu \circ F_\mu^{-1}(B(.6r_h))$. Thus, elliptic regularity applied to the g-harmonic function $F_\mu \circ F_\nu^{-1}$ yields uniform $W^{2,p}$ bounds over an open neighborhood of $B(.6r_h) \cap F_\nu \circ F_\mu^{-1}(B(.6r_h))$, which in turn allows one to construct such a partition of unity.

By [7, Theorem 2.2], for $-2 \le s \le 2$ the Sobolev space defined on M by using local coordinates and the above partition of unity,

$$\|f\|_{H^s(M)} = \sum_{\nu=1}^N \|(f \circ F_\nu^{-1}) \cdot \chi_\nu\|_{H^s(\mathbb{R}^d)},$$

is the same as the space defined intrinsically on (M,g) via the norm

$$\|f\|_{H^s(M)} = \|(1-\Delta_{\mathrm{g}})^{s/2} f\|_{L^2(M)}.$$

The latter is expressed using the expansion of f in an orthonormal eigenfunction basis for Δ_{g}. Furthermore, the two norms are equivalent up to constants depending only on K, D, V, d, which follows by the local equivalence, proven in [7, Theorem 2.2], and the bound on the number N of coordinate charts.

A consequence is the Sobolev embedding theorem for $2 \le q < \infty$, provided that $0 \le s \le 2$,

$$\|f\|_{L^q(M)} \le C(q)\|f\|_{H^s(M)}, \qquad \frac{1}{2} - \frac{1}{q} = \frac{s}{d}\,, \qquad q < \infty\,, \qquad 0 \le s \le 2. \tag{3.4}$$

We now turn to the proof of Theorem 2.1. By local equivalence of the Sobolev norms, finite propagation velocity, and the above partition of unity, it suffices to prove that, for Δ_{g} the Laplacian on $\mathbb{R}^d$ associated to a metric g satisfying (3.3), the following holds for some $T = T(K, D, V, d) > 0$,

$$\begin{aligned} &\|u\|_{L^q(\mathbb{R}^d;L^2([0,T]))} + \|u\|_{L^\infty([0,T];H^{s(q)}(\mathbb{R}^d))} + \|\partial_t u\|_{L^\infty([0,T];H^{s(q)-1}(\mathbb{R}^d))} \\ &\qquad \le C(q)\Big(\|f\|_{H^{s(q)}(\mathbb{R}^d)} + \|g\|_{H^{s(q)-1}(\mathbb{R}^d)} + \|F\|_{L^{q*}(\mathbb{R}^d;L^2([0,T]))}\Big). \end{aligned} \tag{3.5}$$

This was established for $q = \frac{2(d+1)}{d-1}$ in [7, Theorem 2.3] in the case $F = 0$, but as noted there the proof works for $\frac{2(d+1)}{d-1} \le q < \infty$ provided that $s(q) \le 2$. In particular, it holds for $F = 0$ and q in the range of Theorem 2.1. We may also choose a $C(q) = C$ for which (3.5) holds with $F = 0$ for all q in the range of Theorem 2.1, by interpolation. We note that similar arguments show that Theorem 1.2 follows from the Strichartz estimates in local harmonic coordinate charts, which were established (in rescaled form) in [7, Theorem 2.3].

By a duality argument applied to the wave group, the $F = 0$ case of (3.5) for $q = q{*}'$ implies the following energy estimate for the inhomogeneous equation,

$$\begin{aligned} &\|u\|_{L^\infty([0,T];H^{s(q)}(\mathbb{R}^d))} + \|\partial_t u\|_{L^\infty([0,T];H^{s(q)-1}(\mathbb{R}^d))} \\ &\qquad \le C\Big(\|f\|_{H^{s(q)}(\mathbb{R}^d)} + \|g\|_{H^{s(q)-1}(\mathbb{R}^d)} + \|F\|_{L^{q*}(\mathbb{R}^d;L^2([0,T]))}\Big). \end{aligned} \tag{3.6}$$

Here we have used that for q and $q*$ as in Theorem 1.1, we have $s(q) = 1 - s(q{*}')$.

Following [7, Section 2], we introduce a paradifferential approximation P^2 to $-\Delta_{\mathrm{g}}$, where $P = p(x, D)$ is a self-adjoint pseudodifferential operator on $\mathbb{R}^d$ with principal symbol equal to

$$\sum_{k=1}^{\infty}\Big(\sum_{ij} \mathrm{g}_k^{ij}(x)\xi_i\xi_j\Big)^{1/2}\psi_k(\xi).$$

Here, $\mathrm{g}_k^{ij}(x)$ is a mollification of g^{ij} on the spatial scale $2^{-k/2}$, and ψ_k is a Littlewood-Paley partition of unity. Then $|p(x,\xi) - |\xi|| \le c_0|\xi|$, and as a result of [7, (2.7)] we have

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \le C(\alpha,\beta)(1+|\xi|)^{1-|\alpha|+\frac{1}{2}\max(0,|\beta|-1)}, \tag{3.7}$$

or equivalently $p, \partial_x p \in S^1_{1,\frac{1}{2}}$. Also, for g satisfying (3.3), it is shown in [7, Lemma 2.4] that

$$\|P^2 f + \Delta_{\mathrm{g}} f\|_{H^{s-1}(\mathbb{R}^d)} \le C\,\|f\|_{H^s(\mathbb{R}^d)}, \qquad 0 \le s \le 2. \tag{3.8}$$

We now decompose $u = v + w$, where v has vanishing Cauchy data at $t = 0$, and

$$\begin{aligned} (\partial_t^2 + P^2)v(t,x) &= (P^2 + \Delta_{\mathrm{g}})u(t,x), \\ (\partial_t^2 + P^2)w(t,x) &= F(t,x). \end{aligned}$$

By using the Duhamel formula and the bounds (3.6) and (3.8) to estimate v, the bound (3.5) will follow from proving the following estimate,

$$\|w\|_{L^q(\mathbb{R}^d;L^2([0,T]))} \le C\Big(\|f\|_{H^{s(q)}(\mathbb{R}^d)} + \|g\|_{H^{s(q)-1}(\mathbb{R}^d)} + \|(\partial_t^2+P^2)w\|_{L^{q*}(\mathbb{R}^d;L^2([0,T]))}\Big).$$

Let $E(t)$ denote the unitary group $\exp(itP)$ for $t\in\mathbb{R}$. Then as in [7, Section 2] the preceding estimate reduces to the following mapping properties for $E(t)$,

$$\begin{gathered}\|\phi(t)\langle D\rangle^{-s(q)}E(t)f\|_{L^q_xL^2_t(\mathbb{R}^{d+1})} \le C(q)\,\|f\|_{L^2(\mathbb{R}^d)},\\ \|\phi(t)\langle D\rangle^{-s(q)}\int_0^t E(t-s)\phi(s)F(s,\cdot)\,ds\|_{L^q_xL^2_t(\mathbb{R}^{d+1})} \le C(q)\,\|\langle D\rangle^{1-s(q)}F\|_{L^{q*}_xL^2_t(\mathbb{R}^{d+1})},\end{gathered}\tag{3.9}$$

for a $\phi(t)\in C_c^\infty((.3T,.7T))$ that equals 1 on $[\frac13 T,\frac23 T]$.

The first estimate in (3.9) is [7, (2-15)], where it was established for $\frac{2(d+1)}{d-1}\le q<\infty$, following the method used in [11] for smooth metrics.

The proof of the second estimate in (3.9) is more complicated than the situation in [7] and [11], since the integral is over $s<t$, hence the operator of concern cannot be related to the homogeneous wave group using the TT^* method. We will however follow the same idea of [11] in microlocalizing in frequency to a cone along a given spatial direction, and treating that direction as the time variable. The estimate is then obtained by interpolating between energy type estimates and dispersive estimates.

By Littlewood-Paley theory in the spatial variable, and a finite conic partition of unity in the spatial frequency variable, it suffices to consider $E_k^{e_1}(t)=E(t)a_k^{e_1}(D)$, where $a_k^{e_1}(\xi)$ is the product of a homogeneous cutoff to a small cone about the ξ_1 axis with a smooth radial cutoff supported in the shell $2^{k-1}\le|\xi|\le 2^{k+1}$, and prove that

$$\Big\|\phi(t)\int_0^t E_k^{e_1}(t-s)\phi(s)F(s,\cdot)\,ds\Big\|_{L^q_xL^2_t(\mathbb{R}^{d+1})} \le C(q)\,2^k\,\|F\|_{L^{q*}_xL^2_t(\mathbb{R}^{d+1})}.\tag{3.10}$$

By [7, Corollary 6.7], we can write

$$E_k^{e_1}(t,x,y)=\tilde E_k^{e_1}(t,x,y)+R(t),$$

where the operator $\tilde E_k^{e_1}(t)$ is frequency localized on both sides to the $\lambda/100$ neighborhood of $\mathrm{supp}(a_k^{e_1})$, and for all N

$$\sup_{t\in[0,T]}\|R(t)f\|_{H^N}\le C(N)\,2^{-kN}\|f\|_{H^{-N}}.\tag{3.11}$$

Estimate (3.10) holds for $R(t)$ by Sobolev embedding, so it suffices to prove (3.10) with $E_k^{e_1}(t)$ replaced by $\tilde E_k^{e_1}(t)$.

Let $\tilde K_k^{e_1}(t,x,y)$ be the integral kernel of $\tilde E_k^{e_1}(t)$. We will obtain (3.10) from the following pair of bounds,

$$\begin{aligned}\Big\|\phi(t)\int_0^t \tilde K_k^{e_1}(t-s,x_1,x',y_1,y')\phi(s)F(s,y')\,ds\,dy'\Big\|_{L^\infty_{x'}L^2_t(\mathbb{R}^d)}&\\ \le C\,2^{k(d-1)}(1+2^k|x_1-y_1|)^{-\frac{d-1}{2}}\,\|F\|_{L^1_{y'}L^2_s(\mathbb{R}^d)},&\end{aligned}\tag{3.12}$$

and

$$\Big\|\phi(t)\int_0^t \tilde K_k^{e_1}(t-s,x_1,x',y_1,y')\phi(s)F(s,y')\,ds\,dy'\Big\|_{L^2_{x'}L^2_t(\mathbb{R}^d)} \le C\,\|F\|_{L^2_{y'}L^2_s(\mathbb{R}^d)}. \tag{3.13}$$

Interpolation of (3.12) and (3.13) yields

$$\Big\|\phi(t)\int \tilde K_k^{e_1}(t-s,x_1,x',y_1,y')\phi(s)F(s,y_1,y')\,ds\,dy'\Big\|_{L^{q_d}_{x'}L^2_t(\mathbb{R}^d)} \le C\,2^{2ks_d}|x_1-y_1|^{-1+\frac{1}{q_d'}-\frac{1}{q_d}}\|F(\cdot,y_1,\cdot)\|_{L^{q_d'}_{y'}L^2_s(\mathbb{R}^d)},$$

where $q_d=\frac{2(d+1)}{d-1}$, and $s_d=s(q_d)=1/q_d$. The Hardy-Littlewood inequality then yields

$$\Big\|\phi(t)\int_0^t \tilde E_k^{e_1}(t-s)\phi(s)F(s,\cdot)\,ds\Big\|_{L^{q_d}_x L^2_t(\mathbb{R}^{d+1})} \le C\,2^{2ks_d}\,\|F\|_{L^{q_d'}_x L^2_t(\mathbb{R}^{d+1})}.$$

Given $q\ge q_d$ with $q*\le q_d'$, Sobolev embedding now yields (3.10), where we use that

$$\frac{2}{q_d}+d\Big(\frac{1}{q_d}-\frac{1}{q}\Big)+d\Big(\frac{1}{q*}-\frac{1}{q_d'}\Big)=\frac{2}{q_d}+d\Big(\frac{1}{q_d}-\frac{1}{q_d'}\Big)+2=1,$$

and we may assume that F is frequency localized to $|\xi|\approx 2^k$.

By [7, (6.15)] and the comments following it, we have

$$|\tilde K_k^{e_1}(t,x,y)|\le C_N\,2^{kd}(1+2^k|x-y|)^{-\frac{d-1}{2}}\big(1+2^k\big|t-\rho(x,y)\big|\,\big)^{-N},$$

with $\rho(x,y)$ the geodesic distance in g. Estimate (3.12) follows since

$$\|\tilde K_k^{e_1}(t,x,y)\|_{L^1_t([0,T])}\le C\,2^{k(d-1)}(1+2^k|x-y|)^{-\frac{d-1}{2}}.$$

We now turn to (3.13). We first consider the case $|x_1-y_1|\ge 2^{-\frac{k}{2}}$. We will show that for all N we have, uniformly over $0\le t\le T$ and $y_1\in\mathbb{R}$,

$$\sup_{x_1\ge y_1+2^{-k/2}}\Big\|\int \tilde K_k^{e_1}(t,x_1,x',y_1,y')F(y')\,dy'\Big\|_{L^2_{x'}(\mathbb{R}^{d-1})}\le C(N)\,2^{-Nk}\,\|F\|_{L^2(\mathbb{R}^{d-1})}, \tag{3.14}$$

which easily yields (3.13) for $x_1\ge y_1+2^{-\frac{k}{2}}$. This also implies that if $x_1\le y_1-2^{-\frac{k}{2}}$, then

$$\Big\|\phi(t)\int_t^T \tilde K_k^{e_1}(t-s,x_1,x',y_1,y')\phi(s)F(s,y')\,ds\,dy'\Big\|_{L^2_{x'}L^2_t(\mathbb{R}^d)}\le C(N)\,2^{-Nk}\,\|F\|_{L^2(\mathbb{R}^d)}.$$

Thus, (3.13) for $x_1\le y_1-2^{-\frac{k}{2}}$ will follow as a result of the following estimate

$$\Big\|\phi(t)\int_0^T \tilde K_k^{e_1}(t-s,x_1,x',y_1,y')\phi(s)F(s,y')\,ds\,dy'\Big\|_{L^2_{x'}L^2_t(\mathbb{R}^d)}\le C\,\|F\|_{L^2(\mathbb{R}^d)},$$

which was proven in [7, Section 6] using a TT^* argument and the homogeneous energy estimates [7, Theorem 6.9].

Estimate (3.14) results from the fact that bicharacteristics with co-direction in the support of $a_k^{e_1}$ satisfy $\partial_t x_1\approx -1$, and the fact the kernel $\tilde K_k^{e_1}(t,x,y)$ is highly concentrated, in a microlocal sense, within spatial distance $t^{\frac12}2^{-\frac k2}$ of the bicharacteristic flow. We outline the details here for $t\ge 2^{-k}$; the analysis for $0\le t<2^{-k}$ is the same as that for $t=2^{-k}$.

Given $2^{-k}\le t\le T$, and $F\in L^2(\mathbb{R}^{d-1})$, we expand $a_k^{e_1}(D)(F\otimes\delta_{y_1})$ in a rescaled dyadic-parabolic wave packet frame $\{\phi_\gamma\}$ on $\mathbb{R}^d$ as in [7, Section 7]. The elements of this frame are rapidly decreasing outside a ball of diameter $2^{-\frac k2}t^{\frac12}$. Their Fourier transform is supported by ξ in a cone of angle $2^{-\frac k2}t^{-\frac12}$, and where $|\xi|\approx 2^k$. As a result, the coefficients $\{c_\gamma\}$ in the expansion of $a_k^{e_1}(D)(F\otimes\delta_{y_1})$ are highly localized to where the spatial center of the corresponding packet ϕ_γ is distance less than $2^{-\frac k2}t^{\frac12}$ from $x_1=y_1$, and $c_\gamma=0$ unless the Fourier transform of ϕ_γ is supported within a small angle of the ξ_1 axis.

If $\phi_\gamma(y)$ is an element of the frame, centered at (x_γ,ξ_γ), then by [7, Corollary 7.5]

$$|\tilde E_k^{e_1}(t)\phi_\gamma|(x)\le C(N)\,2^{k(\frac{d+1}{2})}t^{-\frac{d-1}{2}}(1+2^{\frac k2}t^{-\frac12}|x-x_t|)^{-N},$$

where x_t is the spatial coordinate of the bicharacteristic through (x_γ,ξ_γ) at $t=0$. In particular, $(x_t)_1\le(x_\gamma)_1-\frac12 t$. If $|(x_\gamma)_1-y_1|\le 2^{-\frac k2}t^{\frac12}$, and $x_1\ge y_1+2^{-\frac k2}$, then

$$1+2^{\frac k2}t^{-\frac12}|x-x_t|\ge\frac18\Big(1+2^{\frac k4}+2^{\frac k2}t^{-\frac12}|x-x_\gamma|\Big)\,.$$

Since the number of indices γ at frequency scale 2^k with center x_γ contained in a given unit ball is bounded by 2^{kd}, estimate (3.14) follows easily since the coefficients in the expansion have ℓ^2 norm bounded by $2^{\frac k2}\|F\|_{L^2}$. For frame elements ϕ_γ with $|(x_\gamma)_1-y_1|\ge 2^{-\frac k2}t^{\frac12}$, a similar argument holds using rapid decrease of the coefficients $\{c_\gamma\}$ away from $(x_\gamma)_1=y_1$.

We now consider (3.13) for $|x_1-y_1|\le 2^{-\frac k2}$, and without loss of generality take $y_1=0$. We will deduce bounds in this region from sideways energy estimates for the operator ∂_t-iP. It suffices to prove (3.13) with $\tilde K_k^{e_1}(t)$ replaced by $K_k^{e_1}(t)$, since the estimate (3.13) is satisfied by the difference $R(t)$. We assume $F(s,y')$ is supported where $.3T\le s\le .7T$, and let

$$u(t,x)=\int_0^t E(t-s)a_k^{e_1}(D)(F(s,\cdot\,)\otimes\delta)\,ds.$$

Note that

$$\|a_k^{e_1}(D)(F(s,\cdot\,)\otimes\delta)\|_{L^2(\mathbb{R}^d)}\le C\,2^{\frac k2}\|F(s,\cdot\,)\|_{L^2(\mathbb{R}^{d-1})},$$

hence by unitarity of $E(t)$ we have an initial bound

$$\|u\|_{L^2_{t,x}([0,T]\times\mathbb{R}^d)}\le C\,2^{\frac k2}\|F\|_{L^2}. \tag{3.15}$$

If $\tilde a_k^{e_1}(\xi)$ equals 1 on a $\lambda/10$ neighborhood of $\mathrm{supp}(a_k^{e_1})$, then since $\tilde a_k^{e_1}(D)\tilde E_k^{e_1}(t)=\tilde E_k^{e_1}(t)$, from the bounds (3.11) for $R(t)$ we have

$$\|(1-\tilde a_k^{e_1}(D))u\|_{L^\infty_t H^N_x([0,T]\times\mathbb{R}^d)}\le C(N)\|F\|_{L^2}. \tag{3.16}$$

Since $\partial_t u=p(x,D)u$ the same holds for u replaced by $\partial_t u$.

Fix $\beta\in C_c^\infty((0,T))$ with $\beta(t)=1$ for $t\in[.1T,.9T]$, and let $v(t,x)=\beta(t)\tilde a_k^{e_1}(D)u(t,x)$. Then

$$(\partial_t-iP)v(t,x)=\partial_t\beta(t)\tilde a_k^{e_1}(D)u(t,x)+i\beta(t)[P,\tilde a_k^{e_1}(D)]u(t,x)+a_k^{e_1}(D)(F\otimes\delta). \tag{3.17}$$

By (3.15) and (3.7),

$$\|\partial_t\beta(t)\tilde a_k^{e_1}(D)u\|_{L^2_{t,x}(\mathbb{R}^{d+1})}+\|\beta(t)[P,\tilde a_k^{e_1}(D)]u\|_{L^2_{t,x}(\mathbb{R}^{d+1})}\le C\,2^{\frac k2}\|F\|_{L^2}.$$

We take the Fourier transform in t of both sides of (3.17), and express the equation as

$$(\tau-p(x,D))\hat v(\tau,x)=\hat G(\tau,x),$$

where we have

$$\|\hat G\|_{L^2_{\tau,x}(\mathbb{R}^{d+1})}\le C\,2^{\frac k2}\|F\|_{L^2}. \tag{3.18}$$

For $\tau\notin[2^{k-1},2^{k+2}]$, the symbol estimates (3.7) and elliptic regularity yield

$$\|D_x\hat v(\tau,\cdot\,)\|_{L^2(\mathbb{R}^d)}\le C\,2^{\frac k2}\|\hat G(\tau,\cdot\,)\|_{L^2(\mathbb{R}^d)}.$$

Since v is supported where $|D_x|\approx 2^k$, we deduce that

$$\|\hat v\|_{L^\infty_{x_1}L^2_{\tau,x'}(\tau\notin[2^{k-1},2^{k+2}])}\le C\,\|F\|_{L^2}.$$

For $\tau\in[2^{k-1},2^{k+2}]$, we can use (3.7) and the fact that $\partial_{\xi_1}p\neq 0$ on $\operatorname{supp}(\tilde a_k^{e_1})$ to factor D_t-P over the support of $\tilde a_k^{e_1}(D)$,

$$\tau-p(x,D)=c(\tau,x,D)(D_1-q(\tau,x,D'))+b(\tau,x,D),$$

where $c,b\in S^0_{1,\frac12}$ with uniform estimates over τ, the symbol c is elliptic, and $q\in S^1_{1,\frac12}$ is self-adjoint in x'. We apply a parametrix for $c(\tau,x,D)$ to write

$$(D_1-q(\tau,x,D'))\hat v(\tau,x)=b_1(\tau,x,D)\hat v(\tau,x)+b_2(\tau,x,D)\hat G(\tau,x), \tag{3.19}$$

where $b_j\in S^0_{1,\frac12}$, uniformly over τ. We may assume that $b_j(\tau,x,\xi)$ is supported in $|\xi|\approx 2^k$, as the remainder is smoothing on v. By self-adjointness of q, we have

$$\|\hat v\|_{L^\infty_{x_1}L^2_{\tau,x'}(\tau\in[2^{k-1},2^{k+2}])}\le C\,\|\hat v\|_{L^2_{\tau,x}}+C\,\|\hat G\|_{L^2_{\tau,x}}\le C\,2^{\frac k2}\|F\|_{L^2},$$

where we use (3.15) and (3.18). Combined with (3.16), we obtain

$$\|u\|_{L^\infty_{x_1}L^2_{t,x}(t\in[.1T,.9T])}\le C\,2^{\frac k2}\|F\|_{L^2}.$$

We now refine the estimate for u over small x_1-intervals. Suppose that $\beta\in C_c^\infty((.1T,.9T))$ with $\beta(t)=1$ for $t\in[.3T,.7T]$. We again let $v=\beta(t)\tilde a_k^{e_1}(D)u$, and repeat the above steps leading to (3.19) for $\tau\in[2^{k-1},2^{k+2}]$. Since the integral kernel of $b_1(\tau,x,D)\tilde a_k^{e_1}(D)$ is dominated by $C(N)\,2^{kd}(1+2^k|x-y|)^{-N}$, we have

$$\|b_1(\tau,x,D)\hat v\|_{L^\infty_{x_1}L^2_{\tau,x'}(\tau\in[2^{k-1},2^{k+2}])}\le C\,\|\widehat{\beta u}\|_{L^\infty_{x_1}L^2_{\tau,x'}}\le C\,2^{\frac k2}\|F\|_{L^2}.$$

Consider now the terms on the right of (3.17), which combine to produce G. From similar bounds on the kernel of $b_2(\tau,x,D)$ and $b_2(\tau,x,D)[P,\tilde a_k^{e_1}(D)]$, we have

$$\|b_2(\tau,x,D)\tilde a_k^{e_1}(D)\widehat{\beta' u}\|_{L^\infty_{x_1}L^2_{\tau,x'}(\tau\in[2^{k-1},2^{k+2}])}\le C\,2^{\frac k2}\|F\|_{L^2},$$

$$\|b_2(\tau,x,D)[P,\tilde a_k^{e_1}(D)]\widehat{\beta u}\|_{L^\infty_{x_1}L^2_{\tau,x'}(\tau\in[2^{k-1},2^{k+2}])}\le C\,2^{\frac k2}\|F\|_{L^2},$$

$$\|b_2(\tau,x,D)a_k^{e_1}(D)(\hat F\otimes\delta)\|_{L^1_{x_1}L^2_{\tau,x'}(\tau\in[2^{k-1},2^{k+2}])}\le C\,\|F\|_{L^2}.$$

Put together, we conclude that for $\tau \in [2^{k-1}, 2^{k+2}]$,

$$\|(D_1 - q(\tau, x, D'))\hat{v}(\tau, x)\|_{L^1_{x_1} L^2_{\tau,x'}(|x_1|<2^{-k/2}, \tau\in[2^{k-1},2^{k+2}])} \le C\,\|F\|_{L^2}.$$

We use (3.14) and (3.11) to deduce that $\|\hat{v}(\tau, 2^{-\frac{k}{2}}, x')\|_{L^2_{\tau,x'}} \le C\|F\|_{L^2}$, and conclude from energy estimates and self-adjointness of q that

$$\|\hat{v}(\tau, x)\|_{L^\infty_{x_1} L^2_{\tau,x'}(|x_1|\le 2^{-k/2})} \le C\,\|F\|_{L^2}.$$

Combined with (3.16) this yields the desired bound for $|x_1| \le 2^{-\frac{k}{2}}$,

$$\|\beta u\|_{L^\infty_{x_1} L^2_{t,x'}(|x_1|\le 2^{-k/2})} \le C\,\|F\|_{L^2},$$

which concludes the proof of (3.13). □

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