ON THE L^p NORM OF SPECTRAL CLUSTERS FOR COMPACT MANIFOLDS WITH BOUNDARY

HART F. SMITH AND CHRISTOPHER D. SOGGE

ABSTRACT. We use microlocal and paradifferential techniques to obtain L^8 norm bounds for spectral clusters associated to elliptic second order operators on two-dimensional manifolds with boundary. The result leads to optimal L^q bounds, in the range $2 \le q \le \infty$, for L^2-normalized spectral clusters on bounded domains in the plane and, more generally, for two-dimensional compact manifolds with boundary. We also establish new sharp L^q estimates in higher dimensions for a range of exponents $\bar{q}_n \le q \le \infty$.

1. INTRODUCTION

Let M be a compact two-dimensional manifold with boundary, and let P be an elliptic, second order differential operator on M, self-adjoint with respect to a density $d\mu$, and with vanishing zeroeth order term, so that in local coordinates

$$(1.1) \qquad \big(Pf\big)(x) = \rho(x)^{-1} \sum_{i,j=1}^{n} \partial_i \Big(\rho(x)\, \mathrm{g}^{ij}(x)\, \partial_j f(x)\Big)\,, \qquad d\mu = \rho(x)\, dx\,.$$

We take g^{ij} to be positive, so that the Dirichlet eigenvalues of P can be written as $\{-\lambda_j^2\}_{j=0}^{\infty}$.

Let χ_λ be the projection of $L^2(d\mu)$ onto the subspace spanned by the Dirichlet eigenfunctions for which $\lambda_j \in [\lambda, \lambda+1]$. In the case that M is compact without boundary of dimension $n \ge 2$, and the coefficients of P are C^∞ functions, Sogge [14] established the following bounds

$$(1.2) \qquad \big\|\chi_\lambda f\big\|_{L^q(M)} \le C\, \lambda^{\frac{n-1}{2}(\frac12 - \frac1q)}\, \|f\|_{L^2(M)}\,, \qquad 2 \le q \le q_n\,.$$

$$(1.3) \qquad \big\|\chi_\lambda f\big\|_{L^q(M)} \le C\, \lambda^{n(\frac12 - \frac1q) - \frac12}\, \|f\|_{L^2(M)}\,, \qquad q_n \le q \le \infty\,.$$

Furthermore, the exponent of λ is sharp on every such manifold (see e.g., [15]). In the case of a sphere, the examples which prove sharpness are in fact eigenfunctions. For (1.2) the appropriate example is an eigenfunction which concentrates in a $\lambda^{-\frac12}$ diameter tube about a geodesic. For (1.3), the example is a zonal eigenfunction of L^2 norm $\lambda^{\frac{n-1}{2}}$ which takes on value comparable to λ on a λ^{-1} diameter ball about each of the north and south poles. Approximate spectral clusters with similar properties can be constructed in the interior of any smooth manifold, showing that for spectral clusters (though not

The authors were supported in part by NSF grants DMS-0140499, DMS-0099642, and DMS-0354668.

necessarily eigenfunctions) the exponents in (1.2) and (1.3) are also lower bounds on manifolds with boundary.

In [13], the authors showed that, on a manifold of dimension $n \geq 2$ for which the boundary is everywhere strictly geodesically concave (such as the complement in $\mathbb{R}^n$ of a strictly convex set) the estimates (1.2) and (1.3) both hold.

On the other hand, Grieser [5] observed that in the unit disk $\{|x| \leq 1\}$ there are eigenfunctions of the Laplacian, for Dirichlet as well as for Neumann boundary conditions, of eigenvalue $-\lambda^2$ that concentrate within a $\lambda^{-\frac{2}{3}}$ neighborhood of the boundary. These are the classical Rayleigh whisphering gallery modes (see [9], [10]). The Fourier-Airy calculus of Melrose and Taylor allows one to construct an approximate spectral cluster with similar localization properties near any boundary point of M at which the boundary is strictly convex (the gliding case). Consequently, if M is of dimension two and the boundary has a point of strict convexity with respect to the metric g (for instance, any smoothly bounded planar domain endowed with the Laplacian and either Dirichlet or Neumann conditions) the following bounds cannot be improved upon

$$\big\|\chi_\lambda f\big\|_{L^q(M)} \leq C\,\lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{q})}\,\|f\|_{L^2(M)}\,, \qquad 2 \leq q \leq 8\,. \tag{1.4}$$

$$\big\|\chi_\lambda f\big\|_{L^q(M)} \leq C\,\lambda^{2(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}\,\|f\|_{L^2(M)}\,, \qquad 8 \leq q \leq \infty\,. \tag{1.5}$$

In this paper we show that the estimates (1.4) and (1.5) hold on any two dimensional compact manifold with boundary, for P as above and either Dirichlet or Neumann conditions assumed. Estimate (1.4) follows by interpolation of the trivial case $q = 2$ with the case $q = 6$, so we restrict attention to $q \geq 6$ for (1.4). For $q \geq 6$, the above estimates are an immediate consequence of the following theorem (see for example [8] or [11]).

Theorem 1.1. *Suppose that u solves the Cauchy problem on $\mathbb{R} \times M$*

$$\partial_t^2 u(t,x) = Pu(t,x)\,, \qquad u(0,x) = f(x)\,, \qquad \partial_t u(0,x) = 0\,, \tag{1.6}$$

and satisfies either Dirichlet conditions

$$u(t,x) = 0 \quad \text{if} \quad x \in \partial M\,,$$

or Neumann conditions, where N_x is a unit normal field with respect to g*,*

$$N_x \cdot \nabla_x u(t,x) = 0 \quad \text{if} \quad x \in \partial M\,.$$

Then the following bounds hold for $6 \leq q \leq 8$,

$$\|u\|_{L^q_x L^2_t(M\times[-1,1])} \leq C\,\|f\|_{H^{\gamma(q)}(M)}\,, \qquad \gamma(q) = \tfrac{2}{3}\big(\tfrac{1}{2}-\tfrac{1}{q}\big)\,,$$

and the following bounds hold for $8 \leq q \leq \infty$,

$$\|u\|_{L^q_x L^2_t(M\times[-1,1])} \leq C\,\|f\|_{H^{\delta(q)}(M)}\,. \qquad \delta(q) = 2\big(\tfrac{1}{2}-\tfrac{1}{q}\big)-\tfrac{1}{2}\,.$$

In the statement of the theorem, the space $H^s(M)$ refers to the Sobolev space of order s on M determined, respectively, by Dirichlet or Neumann eigenfunctions.

Our approach to proving Theorem 1.1 is to work in geodesic normal coordinates near ∂M, and to extend both the operator P and the solution u across the boundary, to obtain u as a solution to a wave equation on an open set, but for an operator with coefficients of Lipschitz regularity. We then adapt a frequency dependent scaling argument, originally developed to handle Lipschitz metrics, to metrics with the particular type of codimension-1 singularities that the extended P will have.

We remark that, for operators of the type (1.1) with ρ and g^{ij} of Lipschitz regularity, the estimate (1.4) is known on the range $2 \le q \le 6$, as established by the first author in [12], along with a weaker version of (1.5) having larger exponent if $q < \infty$. It is not currently known what the sharp exponents are for general Lipschitz P, since the known counterexamples satisfy the estimates (1.5). The estimates for $q = \infty$ were established for eigenfunctions recently by Grieser [6], while the sup-norm estimates for spectral clusters were obtained by the second author in [16].

For $q = \infty$, the squarefunction estimate of Theorem 2.1 below was shown in [12] to hold for operators P with Lipschitz coefficients, which in particular implies the $q = \infty$ case of Theorem 1.1 for P on a manifold with boundary. Our proof here of the case $q < \infty$, however, depends crucially on the fact that if u is appropriately microlocalized away from directions tangent to ∂M, then better squarefunction estimates hold than do for directions near to tangent. In other words, we exploit the fact that the more highly localized eigenfunctions considered in [5] are associated only to gliding directions along ∂M, not directions transverse to ∂M.

A historical curiosity is that the critical $L^2 \to L^8$ bounds for χ_λ have an analog in Euclidean space which seems to be the first restriction theorem for the Fourier transform. To explain this, we first notice that by duality our $L^2 \to L^8$ bounds are equivalent to the statement that $\chi_\lambda : L^{8/7} \to L^2$ with norm $O(\lambda^{1/4})$. The Euclidean analog would say that if $\chi_\lambda : L^{8/7}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ denotes the projection onto Fourier frequencies $|\xi| \in [\lambda, \lambda+1]$, then this operator also has norm $O(\lambda^{1/4})$. An easy scaling argument shows then that the latter result is equivalent to the following Fourier restriction theorem for the circle

$$\Big(\int_0^{2\pi} |\hat f(\cos\theta, \sin\theta)|^2 \, d\theta\Big)^{1/2} \le C\|f\|_{L^{8/7}(\mathbb{R}^2)}, \quad f \in C_0^\infty(\mathbb{R}^2).$$

Stein [17] proved this using a now standard TT^* argument, together with DeLeeuw's [4] observation that $\widehat{d\theta}$ maps $L^{8/7}(\mathbb{R}^2) \to L^8(\mathbb{R}^2)$ by the Hardy-Littlewood-Sobolev theorem, as $|\widehat{d\theta}| \le C|x|^{-1/2}$. Since this argument does not use the oscillations of $\widehat{d\theta}$, one can strengthen the above restriction theorem to show that, for $j \ge 1$, one has the uniform bounds

$$\Big(\int_0^{2^{-j}} |\hat f(\cos\theta, \sin\theta)|^2 \, d\theta\Big)^{1/2} \le C\, 2^{-j/8}\|f\|_{L^{8/7}(\mathbb{R}^2)}, \quad f \in C_0^\infty(\mathbb{R}^2). \tag{1.7}$$

By the Knapp example, there is no small angle improvement for the critical $L^{6/5}(\mathbb{R}^2) \to L^2(\mathbb{S}^1)$ restriction theorem of Stein-Tomas. A key step for us is that in the setting of compact manifolds with boundary we also get the same $O(2^{-j/8})$ improvement in our L^8-estimates when microlocalized to regions of phase space that correspond to bicharacteristics that are of angle comparable to 2^{-j} from tangency to the boundary.

In higher dimensions the natural analog of (1.4)-(1.5) would say that

$$\big\|\chi_\lambda f\big\|_{L^q(M)} \le C\, \lambda^{(\frac23 + \frac{n-2}{2})(\frac12 - \frac1q)}\, \|f\|_{L^2(M)}\,, \qquad 2 \le q \le \tfrac{6n+4}{3n-4}\,. \tag{1.8}$$

$$\big\|\chi_\lambda f\big\|_{L^q(M)} \le C\, \lambda^{n(\frac12 - \frac1q) - \frac12}\, \|f\|_{L^2(M)}\,, \qquad \tfrac{6n+4}{3n-4} \le q \le \infty\,. \tag{1.9}$$

By higher dimensional versions of the Rayleigh whispering gallery modes, this would be sharp if true. At present we are unable to prove this estimate but, as we shall indicate in the final section, we can prove the bounds in (1.9) for the smaller range of exponents $q \geq 4$ if $n \geq 4$, and $q \geq 5$ if $n = 3$. We hope to return to the problem of proving sharp results in higher dimensions in a future work.

Notation. We use the following notation. The symbol $a \lesssim b$ means that $a \leq C\, b$, where C is a constant that depends only on globally fixed parameters (or on N, α, β in case of inequalities involving general integers.)

For convenience we will let x_3 serve as substitute for the time variable t. We use $d = (d_1, d_2, d_3)$ to denote the gradient operator, and $D = -id$.

2. Dyadic Localization Arguments

The estimates of Theorem 1.1 hold if u is supported away from ∂M by the results of [8]. Consequently, by finite propagation velocity and the use of a smooth partition of unity we may assume that, for T small, the solution $u(t,x)$ in Theorem 1.1 is for $|t| \leq T$ supported in a suitably small coordinate patch centered on the boundary. Note that if we establish Theorem 1.1 on the set $|t| \leq T$ for some small T, it then holds for $T = 1$ by energy conservation.

We work in boundary normal coordinates for the Riemannian metric g_{ij} that is dual to g^{ij} of (1.1). Thus, $x_2 > 0$ will define the manifold M, and x_1 is a coordinate function on ∂M which we choose so that ∂_{x_1} is of unit length along ∂M. In these coordinates,

$$\mathrm{g}_{22}(x_1,x_2) = 1\,, \qquad \mathrm{g}_{11}(x_1,0) = 1\,, \qquad \mathrm{g}_{12}(x_1,x_2) = \mathrm{g}_{21}(x_1,x_2) = 0\,. \tag{2.1}$$

The metric g^{ij} for P is the inverse to g_{ij}, and the same equalities hold for it.

We now extend the coefficient g^{11} and ρ in an even manner across the boundary, so that

$$\mathrm{g}^{11}(x_1,-x_2) = \mathrm{g}^{11}(x_1,x_2)\,, \qquad \rho(x_1,-x_2) = \rho(x_1,x_2)\,. \tag{2.2}$$

The extended functions are then piecewise smooth, and of Lipschitz regularity across $x_2 = 0$. Because g is diagonal, the operator P is preserved under the reflection $x_2 \to -x_2$.

After multiplying $\rho(x)$ by a constant, and rescaling variables if necessary, we may assume that on the ball $|x| < 1$ the function $\rho(x)$ is $C^N(\mathbb{R}_+)$ close to the function 1, and $\mathrm{g}^{ij}(x)$ is $C^N(\mathbb{R}_+)$ close to the euclidean metric, where N is suitably large, and c_0 will be taken suitably small,

$$\|\,\rho - 1\|_{C^N(\mathbb{R}^2_+)} \leq c_0\,, \qquad \|\,\mathrm{g}^{ij} - \delta^{ij}\|_{C^N(\mathbb{R}^2_+)} \leq c_0\,. \tag{2.3}$$

We may then extend ρ and g^{ij} globally, preserving conditions (2.1)–(2.3), so that P is defined globally on $\mathbb{R}^2$ and such that

$$\rho(x) = 1\,, \qquad \mathrm{g}^{ij}(x) = \delta^{ij} \quad \text{for } \ |x| \geq \frac{3}{4}\,. \tag{2.4}$$

We then extend the initial data f and the solution u to be odd in x_2 (respectively even in x_2 in case of Neumann conditions). This extension map is seen to map the Dirichlet (respectively Neumann) Sobolev space $H^2(\mathbb{R}^n_+)$ to $H^2(\mathbb{R}^n)$, hence $H^\delta(\mathbb{R}^n_+)$ to $H^\delta(\mathbb{R}^n)$ for $0 \leq \delta \leq 2$. The extended solution u thus solves the extended equation $\partial_t^2 u = Pu$ on

$\mathbb{R}\times\mathbb{R}^2$, with the extended initial data f. The result of Theorem 1.1 is thus a direct consequence of the following

Theorem 2.1. *Suppose that the operator P takes the form* (1.1)*, and that ρ and* g *satisfy conditions* (2.1)–(2.4) *above. Let u solve the Cauchy problem on $\mathbb{R}\times\mathbb{R}^2$*

$$\partial_t^2 u(t,x) = Pu(t,x)\,, \qquad u(0,x)=f(x)\,, \qquad \partial_t u(0,x) = g(x)\,. \tag{2.5}$$

Then the following bounds hold for $6\le q\le 8$,

$$\|u\|_{L^q_xL^2_t(\mathbb{R}^2\times[-1,1])} \lesssim \big(\,\|f\|_{H^{\gamma(q)}}+\|g\|_{H^{\gamma(q)-1}}\big)\,, \qquad \gamma(q)=\tfrac{2}{3}\big(\tfrac12-\tfrac1q\big)\,,$$

and the following bounds hold for $8\le q\le\infty$,

$$\|u\|_{L^q_xL^2_t(\mathbb{R}^2\times[-1,1])} \lesssim \big(\,\|f\|_{H^{\delta(q)}}+\|g\|_{H^{\delta(q)-1}}\big)\,, \qquad \delta(q)=2\big(\tfrac12-\tfrac1q\big)-\tfrac12\,.$$

We begin by reducing matters to compactly supported u satisfying an inhomogeneous equation. Henceforth, we will use notation $x_3=t$. Let $\phi(x)$ be a smooth even function on $\mathbb{R}^3$, equal to 1 for $|x|\le 3/2$, and vanishing for $|x|\ge 2$. We may then write

$$\sum_{j=1}^3 D_j\Big(\,a^{jj}(x)\,D_j(\phi u)(x)\Big)=\sum_{j=1}^3 D_jF_j(x)\,,$$

where

$$a^{33}(x)=\rho(x)\,, \qquad a^{jj}(x)=-\rho(x)\,\mathrm{g}^{jj}(x)\quad\text{for}\;\; j=1,2.$$

We express this equation concisely as $DAD(\phi u)=DF$, and observe that for $0\le\delta\le 2$

$$\|\phi u\|_{H^\delta(\mathbb{R}^3)}+\|F\|_{H^\delta(\mathbb{R}^3)}\;\lesssim\;\|f\|_{H^\delta}+\|g\|_{H^{\delta-1}}\,.$$

This is a consequence of energy estimates, which hold separately on $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$, together with the fact that $DAD(\phi u)$ is compactly supported and has integral 0, so may be written as DF.

We may thus assume that $u(x)$ is supported in the ball $|x|\le 2$, and need to show that

$$\|u\|_{L^qL^2}\lesssim\|u\|_{H^{\gamma(q)}}+\|F\|_{H^{\gamma(q)}}\,, \qquad 6\le q\le 8\,, \tag{2.6}$$

$$\|u\|_{L^qL^2}\lesssim\|u\|_{H^{\delta(q)}}+\|F\|_{H^{\delta(q)}}\,, \qquad 8\le q\le \infty\,, \tag{2.7}$$

where $DADu=DF$.

Next let $\Gamma(\xi)$ be a multiplier of order 0, supported in the set $\frac14\,|\xi_3|\le|\xi_1,\xi_2|\le 4|\xi_3|$, which equals 1 on the set $\frac12\,|\xi_3|\le|\xi_1,\xi_2|\le 2|\xi_3|$. The operator DAD is elliptic on the support of $1-\Gamma$, and we may write

$$DAD\big(1-\Gamma(D)\big)u=\big(1-\Gamma(D)\big)DF-D\big[A,\Gamma(D)\big]Du\,.$$

As a consequence of the Coifman-Meyer commutator theorem [2] (see also Proposition 3.6.B of [19]) the operator $[A,\Gamma(D)]$ maps $H^{\delta-1}\to H^\delta$ for $0\le\delta\le 1$. Hence, the right hand side of the above belongs to $H^{\delta-1}$, and by Sobolev embedding and elliptic regularity (see, for example, Theorem 2.2.B of [19], which applies in the Sobolev setting) we have

$$\|\big(1-\Gamma(D)\big)u\|_{L^qL^2}\lesssim\|\big(1-\Gamma(D)\big)u\|_{H^{\delta(q)+1}}\lesssim\|u\|_{H^{\delta(q)}}+\|F\|_{H^{\delta(q)}}\,.$$

Indeed, there is an extra $\frac12$ derivative in $\delta(q)+1$ beyond the Sobolev index $n(\frac12-\frac1q)$, so this holds for all $2\le q\le\infty$. Since $\gamma(q)\ge\delta(q)$ for $q\le 8$, this implies that (2.6) and (2.7) hold for u replaced by $(1-\Gamma(D))u$ on the left hand side.

It thus remains to establish (2.6) and (2.7) with u replaced on the left by $\Gamma(D)u$. We take a Littlewood-Paley decomposition in ξ to write

$$\Gamma(D)u = \sum_{k=1}^{\infty} \Gamma_k(D)u = \sum_{k=1}^{\infty} u_k \,,$$

with $\widehat{u_k}$ is supported in a region where $|\xi_1, \xi_2| \approx |\xi_3|$ and $|\xi| \approx 2^k$. Since these regions have finite overlap in the ξ_3 axis, we have

$$\|\Gamma(D)u\|_{L^q L^2} \lesssim \|u_k\|_{L^q \ell^2_k L^2} \lesssim \|u_k\|_{\ell^2_k L^q L^2} \,,$$

where we use $q \geq 2$ at the last step.

Now let A_k denote the matrix of coefficients obtained by truncating the frequencies of $a^{ii}(x)$ to $|\xi| \leq c\, 2^k$ for a fixed small c. We then have $DA_k Du_k = DF_k$, where

$$F_k = \Gamma_k(D)F + \bigl[A, \Gamma_k(D)\bigr]Du + \bigl(A_k - A\bigr)Du_k \,. \tag{2.8}$$

Note that the inhomogeneity F_k is now localized in frequency to $|\xi_3| \approx |\xi| \approx 2^k$, by the frequency localizations of A_k and u_k.

We claim that, for $0 \leq \delta \leq 1$,

$$\sum_{k=1}^{\infty} 2^{2k\delta} \|F_k\|^2_{L^2} \lesssim \|u\|^2_{H^\delta} + \|F\|^2_{H^\delta} \,.$$

This follows by orthogonality for the first term on the right of (2.8), and the last term is handled by the bound $\|A - A_k\|_{L^\infty} \lesssim 2^{-k}$. The middle term is handled by the Coifman-Meyer commutator theorem, which yields that $\sum_{k=1}^{\infty} \varepsilon_k \bigl[A, \Gamma_k(D)\bigr]$ maps $H^{\delta-1} \to H^\delta$ for all sequences $\varepsilon_k = \pm 1$.

We thus are reduced to establishing uniform estimates for each dyadically localized piece u_k. We thus fix a frequency scale $\lambda = 2^k$ for the rest of this paper. We then need to prove the following estimates, where we now set $DA_\lambda Du_\lambda = F_\lambda$,

$$\|u_\lambda\|_{L^q L^2(\mathbb{R}^3)} \lesssim \lambda^{\gamma(q)} \bigl(\|u_\lambda\|_{L^2(\mathbb{R}^3)} + \lambda^{-1} \|F_\lambda\|_{L^2(\mathbb{R}^3)} \bigr) , \quad 6 \leq q \leq 8 \,,$$

$$\|u_\lambda\|_{L^q L^2(\mathbb{R}^3)} \lesssim \lambda^{\delta(q)} \bigl(\|u_\lambda\|_{L^2(\mathbb{R}^3)} + \lambda^{-1} \|F_\lambda\|_{L^2(\mathbb{R}^3)} \bigr) , \quad 8 \leq q \leq \infty \,.$$

Since we are using x_3 orthogonality to make this reduction, we must control the norms of the u_λ globally. However, since u is supported in the ball of radius 2, it is easy to see that the norm of u_λ over $|x| \geq 3$ is bounded by $\lambda^{-1}\|u\|_{L^2}$, so in fact it suffices to establish the above estimate with the left hand side norm taken over the cube of sidelength 3.

If we let v_λ denote the localization of u_λ to frequencies where $|\xi_2| \geq \frac{1}{8}\,|\xi_3|$, then the square function estimates hold for v_λ as on an open manifold,

$$\|v_\lambda\|_{L^q L^2(\mathbb{R}^3)} \lesssim \ \lambda^{\delta(q)} \bigl(\|u_\lambda\|_{L^2(\mathbb{R}^3)} + \|F_\lambda\|_{L^2(\mathbb{R}^3)} \bigr) , \quad 6 \leq q \leq \infty \,.$$

This will follow as a consequence of the techniques we use to handle the part of u_λ with frequencies localized to angle ≈ 1 from the ξ_3 axis.

Consequently, we will assume that

$$\operatorname{supp}\bigl(\widehat{u_\lambda}\bigr) \subseteq \bigl\{ \, \xi \, : \, |\xi_1| \in \bigl[\tfrac{1}{2}\lambda, 2\lambda\bigr] , \quad |\xi_2| \leq \tfrac{1}{10}\lambda , \quad |\xi_3| \in \bigl[\tfrac{1}{2}\lambda, 2\lambda\bigr] \, \bigr\} \,.$$

On this region, the operator $DA_\lambda D$ is hyperbolic with respect to the x_1 direction. We can thus take $p(x,\xi')$ a positive elliptic symbol in $\xi' = (\xi_2,\xi_3)$, so that

$$a^{11}_\lambda(x)\big(\,\xi_1^2 - p(x,\xi')^2\,\big) = \sum_{j=1}^3 a^{jj}_\lambda(x)\xi_j^2 \quad \text{if} \quad |\xi_2| \le \tfrac19\lambda\,, \quad |\xi_3| \in \big[\tfrac13\lambda, 3\lambda\big]\,,$$

and such that

$$p(x,\xi') = |\xi'| \quad \text{if} \quad \xi' \notin \big[-\tfrac18\lambda, \tfrac18\lambda\big] \times \big[\tfrac14\lambda, 4\lambda\big]\,.$$

We also smoothly set $p(x,\xi') = 1$ near $\xi' = 0$. Thus,

$$p(x,\xi')\,,\ d_x p(x,\xi') \in S^1_{1,1}\,,$$

and $p(x,\xi')$ differs from $|\xi'|$ by a symbol supported in the dyadic shell $|\xi'| \approx \lambda$.

Next, let $p_\lambda(x',\xi)$ be obtained by truncating the symbol $p(x,\xi')$ to x'-frequencies less than $c\lambda$, where c is a small constant. Then, uniformly over λ,

$$p_\lambda(x,\xi') - p(x,\xi') \in S^0_{1,1}\,, \quad \text{support}\,(\,p_\lambda - p) \subset \big\{\,\xi' \,:\, |\,\xi'| \approx \lambda\,\big\}\,.$$

Furthermore the symbol-composition rule holds for p_λ to first order. Consequently, we can write

$$\big(\,D_1 + p_\lambda(x,D')\,\big)\big(\,D_1 - p_\lambda(x,D')\,\big)u_\lambda = F'_\lambda\,,$$

where

$$\|F'_\lambda\|_{L^2(\mathbb{R}^3)} \lesssim \lambda\,\|u_\lambda\|_{L^2(\mathbb{R}^3)} + \|F_\lambda\|_{L^2(\mathbb{R}^3)}\,.$$

The function u_λ can be written as the sum of four pieces with disjoint Fourier transforms, according to the possible signs of ξ_1 and ξ_3. We restrict attention to the piece u^+_λ, supported where $\xi_1 > 0$ and $\xi_3 > 0$. Estimates for the other pieces will follow similarly. Since p_λ is x'-frequency localized, F'_λ also splits into four disjoint pieces. The symbol $\xi_1 + p(x,\xi')$ is elliptic on the region $\xi_1 > 0$, hence we may write

$$D_1 u^+_\lambda - p_\lambda(x,D')u^+_\lambda = F''_\lambda\,,$$

where

$$\|F''_\lambda\|_{L^2(\mathbb{R}^3)} \lesssim \|u_\lambda\|_{L^2(\mathbb{R}^3)} + \lambda^{-1}\|F_\lambda\|_{L^2(\mathbb{R}^3)}\,.$$

Finally, we have that

$$p_\lambda(x,D') - p_\lambda(x,D')^* \in \mathrm{Op}(S^0_{1,1})\,,$$

and is dyadically supported in ξ'. We have thus reduced the proof of Theorem 2.1 to the following.

Theorem 2.2. *Suppose that the x'-Fourier transform of u_λ satisfies the support condition*

$$\mathrm{supp}\big(\widehat{u_\lambda}\big) \subseteq \big\{\,\xi' \,:\, |\xi_2| \le \tfrac{1}{10}\lambda\,, \quad \xi_3 \in \big[\tfrac12\lambda, 2\lambda\big]\,\big\}\,,$$

and that

$$D_1 u_\lambda - P_\lambda u_\lambda = F_\lambda\,,$$

where $P_\lambda = \frac12\big(\,p_\lambda(x,D') + p_\lambda(x,D')^*\big)$. Then, for $S = [0,1]\times\mathbb{R}^2$,

$$\|u_\lambda\|_{L^qL^2(S)} \lesssim \lambda^{\gamma(q)}\big(\,\|u_\lambda\|_{L^\infty L^2(S)} + \|F_\lambda\|_{L^2(S)}\,\big)\,, \quad 6 \le q \le 8\,,$$

(2.9)

$$\|u_\lambda\|_{L^qL^2(S)} \lesssim \lambda^{\delta(q)}\big(\,\|u_\lambda\|_{L^\infty L^2(S)} + \|F_\lambda\|_{L^2(S)}\,\big)\,, \quad 8 \le q \le \infty\,.$$

The use of the $L^\infty L^2$ norm of u_λ is allowed by Duhamel and energy bounds. Here, as in what follows, we are using the shorthand mixed-norm notation that $L^pL^q = L^p_{x_1}L^q_{x'}$.

3. The Angular Localization

In this section we take a further decomposition of u_λ, by decomposing its Fourier transform dyadically in the ξ_2 variable. The reductions of the previous section required only the fact that the coefficients $a^{jj}(x)$ were Lipschitz functions. The reduction to estimates for angular pieces depends on the fact that the singularities of $a^{jj}(x)$, and hence the points where the x_2-derivatives of $p_\lambda(x,\xi')$ are large, occur only at $x_2=0$. Consequently, various error terms that arise in this further reduction will be highly concentrated at $x_2=0$, which we express through weighted L^2 estimates.

We will take a dyadic decomposition of the ξ_2 variable, from scale $\xi_2\approx\lambda^{\frac23}$ to $\xi_2\approx\lambda$. Thus, for $1\le j<N_\lambda=\frac13\log_2\lambda$, let $\beta_j(\xi')=\beta_j(\xi_2,\xi_3)$ denote a smooth cutoff satisfying

$$\operatorname{supp}(\beta_j)\subset[2^{-j-2}\lambda,2^{-j+1}\lambda]\times[\tfrac14\lambda,4\lambda]\,,$$

and β_{N_λ} supported in $[-\lambda^{\frac23},\lambda^{\frac23}]\times[\frac14\lambda,4\lambda]$, such that, with $\beta_{-j}(\xi_2,\xi_3)=\beta_j(-\xi_2,\xi_3)$

$$\sum_{j=1}^{N_\lambda}\beta_j(\xi')+\sum_{j=1-N_\lambda}^{-1}\beta_j(\xi')=1\quad\text{if}\quad|\xi_2|\le\tfrac18\lambda\quad\text{and}\quad\xi_3\in[\tfrac12\lambda,2\lambda]\,.$$

Let

$$u_j(x)=\beta_j(D')u_\lambda(x)\,.$$

If we define

$$\theta_j=2^{-|j|}\,,$$

then u_j has frequencies localized to $\xi_2\approx\pm\theta_j\,\xi_3$, or $|\xi_2|\lesssim\lambda^{-\frac13}\xi_3$ in case $j=N_\lambda$.

On the microlocal support of u_j, the bicharacteristic equation for the principal symbol $\xi_1-p_\lambda(x,\xi')$ satisfies $\frac{dx_2}{dx_1}\approx\pm\theta_j$, respectively as $j>0$ or $j<0$. A bicharacteristic curve passing through the microlocal support of u_j will satisfy this condition on an interval of x_1-length less than $\varepsilon\theta_j$, if ε is a small constant. It is thus natural that we will have good estimates for u_j on slabs of width $\varepsilon\theta_j$ in the x_1 variable, and it turns out this is sufficient to prove Theorem 2.2.

In proving estimates for u_j, it is convenient to work with the symbol p_j obtained by truncating $p(x,\xi')$ to x'-frequencies less that $c\,\theta_j^{-\frac12}\lambda^{\frac12}$. This finer truncation than that of p_λ is chosen so that, after rescaling space by θ_j, the rescaled symbol $p_j(\theta_j x,\,\cdot\,)$ will be x'-frequency truncated at $\mu^{\frac12}$, where $\mu=\theta_j\lambda$ is the frequency scale of the rescaled solution $u_j(\theta_j x)$. This square root truncation is consistent with the wave packet techniques we use, and is standard in the construction of parametrices for rough metrics.

The energy of the induced error term $(P_\lambda-P_j)u$ will be large at $x_2=0$, but decays away from $x_2=0$ at a rate that is integrable along bicharacteristic curves that traverse the boundary at angle θ_j. This error term can thus be considered as a bounded driving force, and we call this term G_j below.

In the next two sections we will establish the following result.

Theorem 3.1. *Let $S_{j,k}$ denote the slab $x_1\in[k\varepsilon\theta_j,(k+1)\,\varepsilon\theta_j]$, for $0\le k\le\varepsilon^{-1}2^{|j|}$.*

Then, if

$$D_1u_j-P_ju_j=F_j+G_j\,,$$

it holds uniformly over j and k, and $6 \le q \le \infty$, that

$$\|u_j\|_{L^qL^2(S_{j,k})} \lesssim \lambda^{\delta(q)}\theta_j^{\frac12-\frac3q}\Big(\|u_j\|_{L^\infty L^2(S_{j,k})} + \|F_j\|_{L^1L^2(S_{j,k})} + \lambda^{\frac14}\theta_j^{\frac14}\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^{-1}u_j\|_{L^2(S_{j,k})} + \lambda^{-\frac14}\theta_j^{-\frac14}\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^{2}G_j\|_{L^2(S_{j,k})}\Big).$$

For $j = N_\lambda$, it holds that

$$\|u_j\|_{L^qL^2(S_{j,k})} \lesssim \lambda^{\delta(q)}\theta_j^{\frac12-\frac3q}\Big(\|u_j\|_{L^\infty L^2(S_{j,k})} + \|F_j+G_j\|_{L^1L^2(S_{j,k})}\Big).$$

The gain of the factor $\theta_j^{\frac12-\frac3q}$ reflects the fact that, for $q>6$, there is an improvement in the squarefunction estimates if the solution is localized to a small conic set in frequency.

The terms G_j arise naturally in both the linearization step of Lemma 4.4 and the paradifferential smoothing (6.2). They reflect the fact that the singularities of $d^2a^{jj}(x)$ are localized to $x_2=0$. The weighted L^2 bound on u_j is a characteristic energy estimate.

If $\theta_j \approx 1$, then the weighted L^2 bound on G_j dominates the $L^1_{x_2}L^2_{x_1,x_3}$ norm of G_j, and exchanging x_1 and x_2 we could treat G_j and F_j the same. In this case the bound on u_j would be dominated by the $L^\infty_{x_2}L^2_{x_1,x_3}$ norm. For small θ_j, however, we cannot use x_2 as our "time" variable, and we are forced to work with the weighted L^2 norms. These weighted norms can be thought of as an energy norm along the bicharacteristic flow at angle θ_j. Precisely, if one replaced $x_2=\theta_j(x_1-c)$ in the weight, then the weighted L^2 norms of u_j and G_j would behave like the $L^\infty L^2$ and L^1L^2 norms respectively. The crossing point c differs, however, for different bicharacteristics.

The proof of Theorem 3.1 is contained in sections 4 and 5. In section 6 we establish the appropriate bounds on the norms occuring on the right side if, as above, $u_j=\beta_j(D')u_\lambda$, while F_j and G_j are defined in (6.1)-(6.2) below.

To state the bounds required, let $c_{j,k}$ denote the term occuring inside parentheses on the right hand side of Theorem 3.1. In section 6, we show that, if $D_1u_\lambda - P_\lambda u_\lambda = F_\lambda$, then we have a uniform summability condition

$$\sum_j c^2_{j,k(j)} \lesssim \|u_\lambda\|^2_{L^\infty L^2(S)} + \|F_\lambda\|^2_{L^2(S)}, \tag{3.1}$$

where $k(j)$ denotes any sequence of values for k such that the slabs $S_{j,k(j)}$ are nested, in that for $j>0$ we have $S_{j+1,k(j+1)}\subset S_{j,k(j)}$ (with the analogous condition for $j<0$.)

In the remainder of this section we show how Theorem 2.2 follows from Theorem 3.1 together with the bound (3.1).

We first remark that, if q is a fixed index with $q\neq 8$, the bounds of Theorem 3.1 hold (with constant depending on q) under the weaker assumption that the $c_{j,k}$ are uniformly bounded by the right side of (3.1). To see this, we sum over the $2^j\varepsilon^{-1}$ slabs and write

$$\|u_j\|_{L^qL^2(S)} \le \Big(\sum_{k=1}^{2^j\varepsilon^{-1}}\|u_j\|^q_{L^qL^2(S_{j,k})}\Big)^{\frac1q} \lesssim \lambda^{\delta(q)}\theta_j^{\frac12-\frac4q}\|c_{j,k}\|_{\ell^\infty_j\ell^\infty_k} \lesssim \lambda^{\delta(q)}\theta_j^{\frac12-\frac4q}\big(\|u_\lambda\|_{L^\infty L^2(S)}+\|F_\lambda\|_{L^2(S)}\big).$$

The values of $\theta_j = 2^{-|j|}$ vary dyadically from $\lambda^{-\frac{1}{3}}$ to 1. For $q > 8$ we can sum over j to obtain

$$\|u_\lambda\|_{L^q L^2(S)} \lesssim \lambda^{\delta(q)} \big(\|u_\lambda\|_{L^\infty L^2(S)} + \|F_\lambda\|_{L^2(S)} \big) ,$$

and for $6 \le q < 8$ the sum yields

$$\|u_\lambda\|_{L^q L^2(S)} \lesssim \lambda^{\delta(q) - \frac{1}{3}(\frac{1}{2} - \frac{4}{q})} \big(\|u_\lambda\|_{L^\infty L^2(S)} + \|F_\lambda\|_{L^2(S)} \big) .$$

The above exponent of λ equals $\gamma(q)$, yielding the desired bound. The geometric sum, however, increases as $q \to 8$, and yields a logarithmic loss in λ at $q = 8$.

To obtain the bound at $q = 8$, and hence uniform bounds over q in Theorem 3.1, we use the following worst-case branching argument. We consider terms with $j > 0$ here, the negative terms being controlled by the same argument.

Let $S_{1,k(1)}$ denote the slab at scale $\varepsilon\, 2^{-1}$ that maximizes $\|u_\lambda\|_{L^8 L^2(S_{1,k})}$. Since the decomposition of u_λ into u_j is a Littlewood-Paley decomposition in the ξ_2 variable, we have

$$\|u_\lambda\|^2_{L^8 L^2(S_{1,k(1)})} \lesssim \Big\| \Big(\sum_{j=1}^{N_\lambda} |u_j|^2 \Big)^{\frac{1}{2}} \Big\|^2_{L^8 L^2(S_{1,k(1)})} .$$

By the Minkowski inequality,

$$\begin{aligned} \Big\| \Big(\sum_{j=1}^{N_\lambda} |u_j|^2 \Big)^{\frac{1}{2}} \Big\|^2_{L^8 L^2(S_{1,k(1)})} &\le \|u_1\|^2_{L^8 L^2(S_{1,k(1)})} + \Big(\sum_{S_{2,k} \subset S_{1,k(1)}} \Big\| \Big(\sum_{j=2}^{N_\lambda} |u_j|^2 \Big)^{\frac{1}{2}} \Big\|^8_{L^8 L^2(S_{2,k})} \Big)^{\frac{2}{8}} \\ &\le \|u_1\|^2_{L^8 L^2(S_{1,k(1)})} + 2^{\frac{2}{8}} \Big\| \Big(\sum_{j=2}^{N_\lambda} |u_j|^2 \Big)^{\frac{1}{2}} \Big\|^2_{L^8 L^2(S_{2,k(2)})} \end{aligned}$$

where $k(2)$ is chosen to maximize $\big\| \big(\sum_{j=2}^\infty |u_j|^2 \big)^{\frac{1}{2}} \|_{L^8(S_{2,k})}$ among the two slabs $S_{2,k}$ contained in $S_{1,k(1)}$. Repeating this procedure yields a nested sequence such that

$$\begin{aligned} \varepsilon^{\frac{1}{4}} \|u_\lambda\|^2_{L^8 L^2(S)} &\le \|u_1\|^2_{L^8 L^2(S_{1,k(1)})} + 2^{\frac{2}{8}} \|u_2\|^2_{L^8 L^2(S_{2,k(2)})} + 2^{\frac{4}{8}} \|u_3\|^2_{L^8 L^2(S_{3,k(3)})} + \cdots \\ &\lesssim \lambda^{2\delta(8)} \big(c^2_{1,k(1)} + c^2_{2,k(2)} + c^2_{3,k(3)} + \cdots \big) \end{aligned}$$

where the last holds by Theorem 3.1 since $\theta_j^{\frac{1}{2} - \frac{3}{8}} = 2^{-\frac{j}{8}}$. □

4. The Wave Packet Transform

The purpose of this section and the next is to establish Theorem 3.1. We assume for these two sections that we have fixed λ and θ_j, and consider $j > 0$ so that $\xi_2 > 0$ (except for the term $j = N_\lambda$, where $|\xi_2| \le \lambda^{\frac{2}{3}}$.)

We will rescale space by θ_j. Thus, we work with the function

$$u(x) = u_j(\theta_j x) ,$$

which for $j \ne N_\lambda$ is supported in the set

$$\xi_2 \in \big[\tfrac{1}{4} \theta_j \mu, 2\theta_j \mu \big] , \qquad \xi_3 \in \big[\tfrac{1}{4} \mu, 4\mu \big] ,$$

where

$$\mu = \theta_j \lambda$$

is now the frequency scale for $u(x)$. For $j = N_\lambda$, we have $|\xi_2| \le \mu^{\frac12}$, and $\theta_{N_\lambda} = \mu^{-\frac12}$.

Let $q(x,\xi')$ denote the rescaled symbol

$$q(x,\xi') = \theta_j p_j(\theta_j x, \theta_j^{-1}\xi')\,,$$

which is truncated to x'-frequencies less than $c\mu^{\frac12}$. For $|\xi'| \approx \mu$, the symbol q satisfies the estimates

$$\big|\partial_x^\beta \partial_{\xi'}^\alpha q(x,\xi')\big| \lesssim \begin{cases} \mu^{1-|\alpha|}\,, & |\beta| = 0\,,\\ c_0\,\big(\,1+\mu^{\frac12(|\beta|-1)}\theta_j\,\langle \mu^{\frac12}x_2\rangle^{-N}\,\big)\,\mu^{1-|\alpha|}\,, & |\beta| \ge 1\,. \end{cases} \tag{4.1}$$

This follows from (6.32).

In the remainder of this section and the next, we will drop the index j. The quantities θ and μ are the two relevant parameters for our purposes. After rescaling the estimates of Theorem 3.1, and translating $S_{j,k}$ in x_1 to $x_1 = 0$, we are reduced to establishing the following. Here, S denotes the (x_1,x') slab $[0,\varepsilon]\times\mathbb{R}^2$.

Theorem 4.1. *Suppose that $\widehat{u}(\xi)$ is supported in the set*

$$\xi_2 \in \big[\tfrac14\theta\mu, 2\theta\mu\big]\,,\qquad \xi_3 \in \big[\tfrac14\mu, 4\mu\big]\,,$$

respectively $|\xi_2| \le \mu^{\frac12}$ in case $\theta = \mu^{-\frac12}$. Suppose that u satisfies

$$D_1 u - q(x,D')u = F + G$$

on the slab S, where q satisfies (4.1)*, and is truncated to x'-frequencies less than $c\mu^{\frac12}$. Then the following bounds hold, uniformly over θ and μ, and $6 \le q \le \infty$,*

$$\|u\|_{L^qL^2(S)} \lesssim \mu^{\delta(q)}\theta^{\frac12-\frac3q}\Big(\,\|u\|_{L^\infty L^2(S)} + \|F\|_{L^1L^2(S)} + \mu^{\frac14}\theta^{\frac12}\|\langle\mu^{\frac12}x_2\rangle^{-1}u\|_{L^2(S)} + \mu^{-\frac14}\theta_j^{-\frac12}\|\langle\mu^{\frac12}x_2\rangle^2 G\|_{L^2(S)}\,\Big)\,,$$

and for $\theta = \mu^{-\frac12}$

$$\|u\|_{L^qL^2(S)} \lesssim \mu^{\delta(q)}\theta^{\frac12-\frac3q}\Big(\,\|u\|_{L^\infty L^2(S)} + \|F+G\|_{L^1L^2(S)}\,\Big)\,.$$

We introduce the wave-packet transform which will be used to establish Theorem 4.1. This transform is essentially the Cordoba-Fefferman wave packet transform, which was used by Tataru in [18] (and its precedents) to establish Strichartz estimates for low regularity metrics. The main difference is that in our applications we use a Schwartz function with compactly supported Fourier transform, instead of the more standard Gaussian function, as the fundamental wave packet. Our transform will act on the $x' = (x_2,x_3)$ variables.

We use the notion of previous sections: $x = (x_1,x_2,x_3) = (x_1,x')$, where x_3 denotes the variable t.

Fix a real, radial Schwartz function $g(z') \in S(\mathbb{R}^2)$, with $\|g\|_{L^2(\mathbb{R}^2)} = (2\pi)^{-1}$, and assume that its Fourier transform $\widehat{g}(\zeta')$ is supported in the ball $\{|\zeta'| \le c\}$. For $\mu \ge 1$, we define $T_\mu : S'(\mathbb{R}^2) \to C^\infty(\mathbb{R}^4)$ by the rule

$$\big(T_\mu f\big)(x',\xi') = \mu^{\frac12}\int e^{-i\langle\xi',y'-x'\rangle}\,g\big(\mu^{\frac12}(y'-x')\big)\,f(y')\,dy'\,.$$

A simple calculation shows that

$$f(y') = \mu^{\frac{1}{2}} \int e^{i\langle \xi', y'-x'\rangle}\, g\big(\mu^{\frac{1}{2}}(y'-x')\big)\,\big(T_\mu f\big)(x',\xi')\,dx'\,d\xi'\,,$$

so that $T_\mu^* T_\mu = I$. In particular,

$$\|T_\mu f\|_{L^2(\mathbb{R}^4)} = \|f\|_{L^2(\mathbb{R}^2)}\,. \tag{4.2}$$

It will be useful to note that this holds in a more general setting.

Lemma 4.2. *Suppose that $g_{x',\xi'}(y')$ is a family of Schwartz functions on $\mathbb{R}^2$, depending on the parameters x' and ξ', with uniform bounds over x' and ξ' on each Schwartz norm of g. Then the operator*

$$\big(T_\mu f\big)(x',\xi') = \mu^{\frac{1}{2}} \int e^{-i\langle \xi', y'-x'\rangle}\, g_{x',\xi'}\big(\mu^{\frac{1}{2}}(y'-x')\big)\, f(y')\,dy'$$

satisfies the bound

$$\|T_\mu f\|_{L^2(\mathbb{R}^4)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}\,.$$

Proof. T_μ is bounded if and only if T_μ^* is bounded. Since $\|T_\mu^* F\|_2^2 \le \|T_\mu T_\mu^* F\|_2 \|F\|_2$, it suffices to see that $T_\mu T_\mu^*$ is bounded on $L^2(dy'd\xi')$.

The operator $T_\mu T_\mu^*$ is an integral operator with kernel

$$K(y',\eta';x',\xi') = \mu\, e^{i\langle \eta', y'\rangle - i\langle \xi', x'\rangle} \int e^{i\langle \xi'-\eta', z'\rangle} g_{y',\eta'}\big(\mu^{\frac{1}{2}}(z'-y')\big)\, \overline{g_{x',\xi'}\big(\mu^{\frac{1}{2}}(z'-x')\big)}\,dz'\,.$$

A simple integration by parts argument shows that

$$\big|K(y',\eta';x',\xi')\big| \lesssim \big(\,1 + \mu^{-\frac{1}{2}}|\eta'-\xi'| + \mu^{\frac{1}{2}}|y'-x'|\,\big)^{-N}\,,$$

with constants depending only on uniform bounds for a finite collection of seminorms of $g_{x',\xi'}$ depending on N. The $L^2(\mathbb{R}^4)$ boundedness of $T_\mu T_\mu^*$ then follows by Schur's Lemma. □

A corollary of this lemma is that, for N positive or negative,

$$\|\langle \mu^{\frac{1}{2}} x_2\rangle^N T_\mu f\|_{L^2(\mathbb{R}^4)} \lesssim \|\langle \mu^{\frac{1}{2}} x_2\rangle^N f\|_{L^2(\mathbb{R}^2)}\,, \tag{4.3}$$

by considering $g_{x'}(y) = \langle \mu^{\frac{1}{2}} x_2\rangle^N \langle \mu^{\frac{1}{2}} x_2 - y_2\rangle^{-N} g(y')\,.$

Lemma 4.3. *Let $q(x,\xi')$ satisfy the estimates (4.1). Suppose that $|\xi'| \approx \mu$. Then, if $q(y,D_y')^*$ acts on the y' variable, and $y_1 = x_1$, we can write*

$$\Big(\,q(y,D_y')^* - i d_{\xi'} q(x,\xi')\cdot d_{x'} + i d_{x'} q(x,\xi')\cdot d_{\xi'}\,\Big)\Big[e^{i\langle \xi', y'-x'\rangle}\, g\big(\mu^{\frac{1}{2}}(y'-x')\big)\Big]$$
$$= e^{i\langle \xi', y'-x'\rangle}\, g_{x,\xi'}\big(\mu^{\frac{1}{2}}(y'-x')\big)$$

where $g_{x,\xi'}(\,\cdot\,)$ denotes a family of Schwartz functions on $\mathbb{R}^2$ depending on the parameters x and ξ', each of which has Fourier transform supported in the ball of radius $2c$. If $\|\cdot\|$ denotes any of the Schwartz seminorms, we have

$$\|g_{x,\xi'}\| \lesssim 1 + c_0\,\mu^{\frac{1}{2}}\theta\langle \mu^{\frac{1}{2}} x_2\rangle^{-3}\,,$$

where c_0 is the small constant of (2.3).

Proof. Letting $\mathfrak{F}$ denote the Fourier transform with respect to y', we write

$$\mathfrak{F}\circ\Big(q(y,D_y')^* - id_{\xi'}q(x,\xi')\cdot d_{x'} + id_{x'}q(x,\xi')\cdot d_{\xi'}\Big)\Big[e^{i\langle\xi',y'-x'\rangle}\,g\big(\mu^{\frac12}(y'-x')\big)\Big](\eta') \\ = e^{-i\langle\eta',x'\rangle}\,\mu^{-1}\,\widehat{g_{x,\xi'}}\big(\mu^{-\frac12}(\eta'-\xi')\big)\,,$$

where $\widehat{g_{x,\xi'}}(\eta')$ is equal to

$$\int e^{-i\langle\eta',y'\rangle}\Big[\,q(x+\mu^{-\frac12}y',\xi'+\mu^{\frac12}\eta') - q(x,\xi') - d_{x',\xi'}q(x,\xi')\cdot\big(\mu^{-\frac12}y',\mu^{\frac12}\eta'\big)\Big]\,g(y')\,dy' \\ = \int_0^1(1-\sigma)\,\Big[\int e^{-i\langle\eta',y'\rangle}\partial_\sigma^2\Big(\,q\big(x+\sigma\mu^{-\frac12}y',\xi'+\sigma\mu^{\frac12}\eta'\big)\Big)\,g(y')\,dy'\Big]\,d\sigma\,.$$

The spectral restriction on q and g imply that this vanishes for $|\eta'|\ge 2c$. Consequently, it suffices to establish C^∞ bounds in η' for the term in brackets, uniformly over $\sigma\in[0,1]$ and $|\eta'|\le 2c$. Since the effect of differentiating the integrand with respect to η' is innocuous, as the rapid decrease in $g(y')$ counters any polynomial in y', we content ourselves with establishing uniform pointwise bounds on the term in brackets. Note that $|\xi'+\sigma\mu^{\frac12}\eta'|\approx\mu$.

The effect of ∂_σ^2 is to bring out factors of $\mu^{\pm\frac12}$, and to differentiate q twice. If q is differentiated at most once in x', then the bounds

$$|\partial_{x'}\partial_{\xi'}q(x,\xi')|\lesssim 1\,,\qquad |\partial_{\xi'}^2q(x,\xi')|\lesssim\mu^{-1}\,,\quad\text{for}\quad|\xi'|\approx\mu\,,$$

yield bounds of size 1 on the term. If q is differentiated twice in x', then by (4.1) we have the bounds, for $|\xi'|\approx\mu$,

$$\mu^{-1}|\partial_{x'}^2q(x+\sigma\mu^{-\frac12}y',\xi')|\lesssim c_0+c_0\,\mu^{\frac12}\theta\langle\mu^{\frac12}x_2+\sigma y_2\rangle^{-3} \\ \lesssim 1+c_0\,\mu^{\frac12}\theta\langle\mu^{\frac12}x_2\rangle^{-3}\langle y_2\rangle^3\,.$$

The rapid decrease of $g(y')$ absorbs the term $\langle y_2\rangle^3$, leading to the desired bounds. □

We now take the wave packet transform of the solution $u(x)$ with respect to the x' variables, and introduce the notation $\tilde u(x,\xi')=(T_\mu u)(x,\xi')$. The functions $\tilde F(x,\xi')$ and $\tilde G(x,\xi')$ in the next lemma, though, include terms in addition to the transforms of F and G of Theorem 4.1. Let $\tilde S$ denote the (x_1,x',ξ') slab $[0,\varepsilon]\times\mathbb{R}^4=S\times\mathbb{R}^2_{\xi'}$.

Lemma 4.4. *Under the above conditions, we may write*

$$\Big(d_1-d_{\xi'}q(x,\xi')\cdot d_{x'}+d_{x'}q(x,\xi')\cdot d_{\xi'}\Big)\tilde u(x,\xi')=\tilde F(x,\xi')+\tilde G(x,\xi')\,,$$

where

$$(4.4)\quad \|\tilde F\|_{L^1L^2(\tilde S)}+\mu^{-\frac14}\theta^{-\frac12}\|\langle\mu^{\frac12}x_2\rangle^2\tilde G\|_{L^2(\tilde S)} \\ \lesssim\|u\|_{L^\infty L^2(S)}+\|F\|_{L^1L^2(S)}+\mu^{\frac14}\theta^{\frac12}\|\langle\mu^{\frac12}x_2\rangle^{-1}u\|_{L^2(S)}+\mu^{-\frac14}\theta^{-\frac12}\|\langle\mu^{\frac12}x_2\rangle^2G\|_{L^2(S)}\,.$$

Furthermore, $\tilde F$ and $\tilde G$ are supported in a set where $\xi_2\approx\theta\mu$, $\xi_3\approx\mu$.

In case $\theta=\mu^{-\frac12}$, then

$$(4.5)\qquad \|\tilde F+\tilde G\|_{L^1L^2(\tilde S)}\lesssim\|u\|_{L^\infty L^2(S)}+\|F+G\|_{L^1L^2(S)}\,,$$

and $\tilde F+\tilde G$ is supported where $|\xi_2|\lesssim\mu^{\frac12}$ and $\xi_3\approx\mu$.

Proof. Applying T_μ to the equation $D_1 u = F + G + q(x,D')^* u$ yields

$$d_1\tilde u(x,\xi') = i\bigl(T_\mu F\bigr)(x,\xi') + i\bigl(T_\mu G\bigr)(x,\xi') \\ + i\,\mu^{\frac12}\int \overline{q(x_1,y',D'_y)^*\Bigl[\,e^{i\langle\xi',y'-x'\rangle}\,g\bigl(\mu^{\frac12}(y'-x')\bigr)\Bigr]}\,u(x_1,y')\,dy'\,.$$

The terms $T_\mu F$ and $T_\mu G$ satisfy the bounds required of $\tilde F$ and $\tilde G$ respectively, the latter by the estimate (4.3) in the case of (4.4). By Lemma 4.3, we can write the last term as

$$\Bigl(d_{\xi'}q(x,\xi')\cdot d_{x'} - d_{x'}q(x,\xi')\cdot d_{\xi'}\Bigr)\tilde u(x,\xi') \\ + \mu^{\frac12}\int e^{-i\langle\xi',y'-x'\rangle}\,g_{x,\xi'}\bigl(\mu^{\frac12}(y'-x')\bigr)\,u(x_1,y')\,dy'\,.$$

For x_2 such that $\mu^{\frac12}\theta\langle\mu^{\frac12}x_2\rangle^{-3}\le 1$, the latter term is absorbed into $\tilde F$ by Lemmas 4.2 and 4.3. For x_2 such that $\mu^{\frac12}\theta\langle\mu^{\frac12}x_2\rangle^{-3}\ge 1$, the term can be absorbed into $\tilde G$, by (4.3) and Lemma 4.3. Here we use the simple fact that (4.3) holds for operators of the type in Lemma 4.2. Note that if $\theta=\mu^{-\frac12}$ the entire term can be absorbed into $\tilde F$.

The support condition on $\tilde F$ and $\tilde G$ follows from the support condition on $\widehat u$, and the fact that $g_{x,\xi'}$ has Fourier transform supported in the ball of radius $2c$. Alternatively, we may multiply both sides of the equation defining $\tilde F+\tilde G$ by a cutoff supported in the set $\xi_3\approx\mu$, $\xi_2\approx\theta\mu$ (respectively $|\xi_2|\lesssim\mu^{\frac12}$), which equals 1 on the support of $\tilde u$. □

Let $\Theta_{s,r}$ denote the canonical transform on $\mathbb{R}^4_{x',\xi'} = T^*(\mathbb{R}^2_{x'})$ generated by the Hamiltonian flow of q. Thus, $\Theta_{s,r}(x',\xi')=\gamma(s)$, where γ is the integral curve of the vector field

$$d_1 - d_{\xi'}q(x,\xi')\cdot d_{x'} + d_{x'}q(x,\xi')\cdot d_{\xi'}$$

with $\gamma(r)=(x',\xi')$. Note that $\Theta_{s,r}$ is symplectic, thus preserves the measure $dx'\,d\xi'$, hence induces a unitary mapping on $L^2(\mathbb{R}^4)$. Furthermore, $\Theta_{s,r}$ maps a set of the form $\xi_2\approx\theta\xi_3$ to a set of similar form, provided $|s-r|\le 1$. This follows since $|d_{x'}q(x,\xi')|\le c\,\theta\,|\xi'|$ for c a small constant.

We can now write

$$\tilde u(x,\xi') = \tilde u(0,\Theta_{0,x_1}(x',\xi')) + \int_0^{x_1}\tilde F(s,\Theta_{s,x_1}(x',\xi'))\,ds + \int_0^{x_1}\tilde G(s,\Theta_{s,x_1}(x',\xi'))\,ds\,. \tag{4.6}$$

By the preceding comments, for each s the integrands are supported in the flowout under $\Theta_{s,0}$ of the set $\xi_3\approx\mu$, $\xi_2\approx\theta\mu$ (respectively $|\xi_2|\lesssim\mu^{\frac12}$).

Writing $u=T^*_\mu\tilde u$ shows that $u(x)$ can be written as a superposition of functions, each of which is the restriction to $x_1>s$ of the image under T^*_μ of a function invariant under the Hamiltonian flow of q. However, in view of the bounds (4.4), the term $\tilde G$ has large L^1L^2 norm if θ is small. As a result, one cannot directly apply (4.6) to reduce matters to considering estimates for such flow-invariant functions. Nevertheless, we can use an argument from Koch-Tataru [7] together with (4.4) to see that we may indeed reduce consideration to the case that $\tilde u$ is invariant under the flow of q. Precisely, we show here that Theorem 4.1 is a consequence of the following theorem, which will be proven in the next section.

Theorem 4.5. *Suppose that $f \in L^2(\mathbb{R}^4)$ is supported in a set of the form $\xi_3 \approx \mu$, $\xi_2 \approx \theta\mu$, or a set of the form $\xi_3 \approx \mu$, $|\xi_2| \lesssim \mu^{\frac12}$ in case $\theta = \mu^{-\frac12}$.*

Then, if $u = T_\mu^*\big[f\big(\Theta_{0,x_1}(x',\xi')\big)\big]$, we have for $q \geq 6$

$$\|u\|_{L^qL^2(S)} \lesssim \mu^{\delta(q)}\theta^{\frac12-\frac3q}\,\|f\|_{L^2(\mathbb{R}^4)}\,.$$

In the remainder of this section, we demonstrate the reduction of Theorem 4.1 to Theorem 4.5. In the case of $\theta = \mu^{-\frac12}$, it is a simple consequence of (4.5) and (4.6).

For general θ this reduction requires the introduction of the space V_q^2 of functions on $\tilde S$ with bounded 2-variation along the Hamiltonian flow of q. Recall that $\Theta_{r,s}$ preserves the measure $dx'\,d\xi'$. Then, following Koch-Tataru [7] we define

$$\|\tilde u\|_{V_q^2}^2 = \|\tilde u(0,\,\cdot\,)\|_{L^2(\mathbb{R}^4)}^2 + \sup_P \sum_{j\geq 1} \|\tilde u(s_j,\,\cdot\,) - \tilde u(s_{j-1},\Theta_{s_{j-1},s_j}(\,\cdot\,))\|_{L^2(\mathbb{R}^4)}^2\,,$$

where P denotes the family of finite partitions $\{0 = s_0 < s_1 < \ldots < s_m = \varepsilon\}$ of $[0,\varepsilon]$.

By Lemma 6.4 of [7], if $\|\tilde u\|_{V_q^2} < \infty$, we may decompose

$$\tilde u = \sum_{k=1}^\infty c_k\,\tilde u_k\,, \qquad \text{with} \quad \sum_{k=1}^\infty |c_k| \leq \|\tilde u\|_{V_q^2}\,,$$

where each function $\tilde u_k$ is an atom, in the sense that for some partition $\{s_j\}$ in P

$$\tilde u_k(x,\xi') = \sum_{j=1}^m 1_{[s_{j-1},s_j)}(x_1) f_j(\Theta_{0,x_1}(x',\xi'))\,,$$

where, for each $q > 2$, it holds that

$$\Big(\sum_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^4)}^q\Big)^{\frac1q} \leq C_q\,.$$

Note that one may bound $C_q \leq C_6$ for $q \geq 6$, so we may take C_q uniformly bounded, since we work with $q \geq 6$.

We also note that each f_j arising in the atomic decomposition of $\tilde u$ will be supported in the region $\xi_3 \approx \mu$, $\xi_2 \approx \theta\mu$. This follows from the inductive construction of f_j in [7], together with the comments surrounding (4.6).

Consider $u_k = T_\mu^*\tilde u_k$. Then, assuming Theorem 4.5, for $q \geq 6$ we may bound

$$\|u_k\|_{L^qL^2(S)} \leq \Big(\sum_{j=1}^m \|T_\mu^*\big[f_j\big(\Theta_{0,x_1}(\,\cdot\,)\big)\big]\|_{L^qL^2(S)}^q\Big)^{\frac1q} \lesssim \Big(\sum_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^4)}^q\Big)^{\frac1q} \lesssim 1\,.$$

Summing over k yields $\|u\|_{L^qL^2(S)} \lesssim \|\tilde u\|_{V_q^2}$. It thus remains to demonstrate that

$$\|\tilde u\|_{V_q^2} \lesssim \|\tilde u(0,\,\cdot\,)\|_{L^2(\mathbb{R}^4)} + \|\tilde F\|_{L^1L^2(\tilde S)} + \mu^{-\frac14}\theta^{-\frac12}\|\langle\mu^{\frac12}x_2\rangle^2\tilde G\|_{L^2(\tilde S)}\,, \tag{4.7}$$

since by Lemma 4.4 and boundedness of T_μ the right hand side here is dominated by the right hand side in Theorem 4.1.

We use the decomposition (4.6), and note that the V_q^2 norm of the first two terms on the right hand side of (4.6) are easily bounded by the first two terms on the right hand side of (4.7), the latter since

$$\sum_j \Bigl\| \int_0^{s_j} \tilde F(s,\Theta_{s,s_j}(x',\xi'))\,ds - \int_0^{s_{j-1}} \tilde F(s,\Theta_{s,s_j}(x',\xi'))\,ds \Bigr\|^2_{L^2(\mathbb{R}^4)}$$
$$\le \sum_j \left(\int_{s_{j-1}}^{s_j} \|\tilde F(s,\Theta_{s,s_j}(x',\xi'))\|_{L^2(\mathbb{R}^4)}\,ds \right)^2$$
$$= \sum_j \left(\int_{s_{j-1}}^{s_j} \|\tilde F(s,\cdot)\|_{L^2(\mathbb{R}^4)}\,ds \right)^2 \lesssim \|\tilde F\|^2_{L^1L^2(\tilde S)},$$

using the invariance of $dx'd\xi'$ under Θ in the second equality.

We thus reduce to the case that $\tilde F = 0$ and $\tilde u(0,x',\xi') = 0$, and hence by (4.6) that

$$\tilde u(x,\xi') = \int_0^{x_1} \tilde G(s,\Theta_{s,x_1}(x',\xi'))\,ds\,.$$

Note that, by the group property of Θ, we have

$$\|\tilde u(s_j,\,\cdot\,) - \tilde u(s_{j-1},\Theta_{s_{j-1},s_j}(\,\cdot\,))\|^2_{L^2(\mathbb{R}^4)} = \Bigl\| \int_{s_{j-1}}^{s_j} \tilde G(s,\Theta_{s,s_j}(\,\cdot\,))\,ds \Bigr\|^2_{L^2(\mathbb{R}^4)}. \tag{4.8}$$

Given a partition $\{0 = s_0 < s_1 < \ldots < s_m = \varepsilon\}$, we first consider the sum of the quantity (4.8) over those indices j for which $|s_j - s_{j-1}| \le \mu^{-\frac12}\theta^{-1}$. By the Schwarz inequality we may bound the sum by

$$\sum_j \theta^{-1}\mu^{-\frac12} \|\tilde G(s,\Theta_{s,s_j}(x',\xi'))\|^2_{L^2([s_{j-1},s_j]\times\mathbb{R}^4)} \le \mu^{-\frac12}\theta^{-1}\|\tilde G\|^2_{L^2(\tilde S)}\,.$$

Next, consider an index j for which $|s_j - s_{j-1}| > \mu^{-\frac12}\theta^{-1}$. We split the interval $[s_j, s_{j-1}]$ into a union of intervals I_k for which $\frac12|I_k| \le \mu^{-\frac12}\theta^{-1} \le |I_k|$. We claim that we may bound

$$\Bigl\| \int_{s_{j-1}}^{s_j} \tilde G(s,\Theta_{s,s_j}(\,\cdot\,))\,ds \Bigr\|^2_{L^2(\mathbb{R}^4)} \lesssim \sum_k \Bigl\| \langle \mu^{\frac12} x_2\rangle^2 \int_{I_k} \tilde G(s,\Theta_{s,s_k}(\,\cdot\,))\,ds \Bigr\|^2_{L^2(\mathbb{R}^4)}, \tag{4.9}$$

where s_k denotes the right endpoint of I_k. Given (4.9), we may apply the Schwarz inequality as before (together with the fact that the weight $\langle \mu^{\frac12} x_2\rangle^2$ is essentially preserved by Θ_{s,s_k}, since $\frac{dx_2}{dx_1} \approx \theta$ on the domain of integration and $|s - s_k| \le \mu^{-\frac12}\theta^{-1}$) to bound the sum over k and then j by the right hand side of (4.7).

To prove (4.9), we write

$$\int_{s_{j-1}}^{s_j} \tilde G(s,\Theta_{s,s_j}(x',\xi'))\,ds = \sum_k \tilde v_k(\Theta_{s_k,s_j}(x',\xi'))\,,$$

with

$$\tilde v_k(x',\xi') = \int_{I_k} \tilde G(s,\Theta_{s,s_k}(x',\xi'))\,ds\,.$$

Then (4.9) will follow by showing that

$$\Big| \int \tilde{v}_k(\Theta_{s_k,s_j}(x',\xi'))\,\overline{\tilde{v}_{k'}(\Theta_{s_{k'},s_j}(x',\xi'))}\,dx'\,d\xi' \Big|$$

$$= \Big| \int \tilde{v}_k(\Theta_{s_k,s_{k'}}(x',\xi'))\,\overline{\tilde{v}_{k'}(x',\xi')}\,dx'\,d\xi' \Big|$$

$$\lesssim |k-k'|^{-2}\,\|\langle \mu^{\frac12}x_2\rangle^2 \tilde{v}_k\|_{L^2(\mathbb{R}^4)}\|\langle \mu^{\frac12}x_2\rangle^2 \tilde{v}_{k'}\|_{L^2(\mathbb{R}^4)}\,.$$

This, in turn, is a simple consequence of the fact that $\frac{dx_2}{dx_1} \approx \theta$ on the domain of integration, and hence, letting x_2 denote the x_2-coordinate function,

$$|\,x_2(\Theta_{s_k,s_{k'}}(x',\xi')) - x_2| \approx \theta\,|s_k - s_{k'}| \approx \mu^{-\frac12}|k-k'|\,.$$

Consequently,

$$\langle \mu^{\frac12}x_2(\Theta_{s_k,s_{k'}}(x',\xi'))\rangle^{-2}\langle \mu^{\frac12}x_2\rangle^{-2} \lesssim |k-k'|^{-2}\,. \quad \square$$

5. Homogeneous Estimates

In this section we prove Theorem 4.5. For notational convenience, the variables $z = (z_2, z_3)$ and $\zeta = (\zeta_2,\zeta_3)$ will be used as dummy variables in the role of x' and ξ', as will w and η. We also use real variables r, s, t as dummy variables in the role of x_1 and y_1. For $f \in L^2(dx'\,d\xi')$, define Wf by the rule

$$Wf(x) = T_\mu^*\big(f\circ\Theta_{0,x_1}\big)(x')\,.$$

Let $\beta_\theta(\xi')$ be a cutoff to the region $\xi_3 \approx \mu$, $\xi_2 \approx \theta\mu$, (respectively $|\xi_2| \le \mu^{\frac12}$ in case $\theta = \mu^{-\frac12}$). Then Theorem 4.5 is equivalent to establishing the bound

$$\|\beta_\theta(D')Wf\|_{L^qL^2(S)} \lesssim \mu^{\delta(q)}\theta^{\frac12-\frac3q}\|f\|_{L^2(\mathbb{R}^4)}\,,$$

which is equivalent to the bound

$$\|\beta_\theta(D')WW^*\beta_\theta(D')F\|_{L^qL^2(S)} \lesssim \mu^{2\delta(q)}\theta^{1-\frac6q}\|F\|_{L^{q'}L^2(S)}\,. \tag{5.1}$$

The operator WW^* takes the form

$$\big(WW^*F\big)(x) = \int_0^\varepsilon T_\mu^*\big[\big(T_\mu F\big)(s,\,\cdot\,)\circ\Theta_{s,x_1}\big](x')\,ds\,.$$

If applied to functions truncated by $\beta_\theta(D')$, then WW^* may be replaced by the integral kernel

$$K(r,x';s,y') = \mu\int e^{i\langle\zeta,x'-z\rangle - i\langle\zeta_{s,r},y'-z_{s,r}\rangle} g\big(\mu^{\frac12}(x'-z)\big)\,g\big(\mu^{\frac12}(y'-z_{s,r})\big)\,\beta_\theta(\zeta)\,dz\,d\zeta\,,$$

where we use the shorthand notation

$$(z_{s,r},\zeta_{s,r}) = \Theta_{s,r}(z,\zeta)\,. \tag{5.2}$$

The factors $\beta_\theta(D')$ in (5.1) can now be ignored (since they are bounded in the desired norms), and we are reduced to establishing mapping properties for K. We observe that (5.1), and hence Theorem 4.5, can be reduced to establishing the following pair of bounds:

$$\sup_{r,s\in[0,\varepsilon]}\Big\|\int K(r,x';s,y')\,f(y')\,dy'\Big\|_{L^2(\mathbb{R}^2)} \le \|f\|_{L^2(\mathbb{R}^2)}\,, \tag{5.3}$$

and

$$\Big\| \int K(r,x';s,y')\,f(y')\,dy\Big\|_{L^\infty_{x_2}L^2_{x_3}(\mathbb{R}^2)} \lesssim \mu\theta\,\big(\,1+\mu\theta^2\,|r-s|\,\big)^{-\frac12}\,\|f\|_{L^1_{y_2}L^2_{y_3}(\mathbb{R}^2)}\,. \tag{5.4}$$

To see this, note that interpolation yields the bound

$$\Big\| \int K(r,x';s,y')\,f(y')\,dy\Big\|_{L^q_{x_2}L^2_{x_3}(\mathbb{R}^2)} \lesssim \big(\mu\theta\big)^{1-\frac2q}\,\big(\,1+\mu\theta^2\,|r-s|\,\big)^{\frac1q-\frac12}\,\|f\|_{L^{q'}_{y_2}L^2_{y_3}(\mathbb{R}^2)}\,.$$

By the convolution property $L^{\frac q2} * L^{q'} \subset L^q$, and the bound

$$\big(\mu\theta\big)^{1-\frac2q}\big\|\big(\,1+\mu\theta^2\,|s|\,\big)^{\frac1q-\frac12}\big\|_{L^{\frac q2}(ds)} \approx \mu^{1-\frac4q}\theta^{1-\frac6q} = \mu^{2\delta(q)}\theta^{1-\frac6q}\,,$$

we obtain the following bound equivalent to (5.1)

$$\Big\| \int K(r,x';s,y')\,F(s,y')\,ds\,dy'\Big\|_{L^q_rL^q_{x_2}L^2_{x_3}(S)} \lesssim \mu^{2\delta(q)}\theta^{1-\frac6q}\,\|F\|_{L^{q'}_sL^{q'}_{y_2}L^2_{y_3}(S)}\,.$$

The bound (5.3) follows immediately from the L^2 boundedness of T_μ, and the fact that $\Theta_{s,r}$ preserves the measure $dz\,d\zeta$, so it remains to establish (5.4). We start by estimating the derivatives of the Hamiltonian flow with respect to the initial parameters z and ζ. We only need bounds for curves lying entirely in the region

$$\zeta_3\in[\tfrac14\mu,2\mu] \quad\text{and}\quad \zeta_2\in[\tfrac14\theta\mu,2\theta\mu]$$

(respectively $|\zeta_2|\le\mu^{\frac12}$ in case $\theta=\mu^{-\frac12}$). In order to avoid extraneous powers of μ it is convenient to exploit homogeneity to reduce to the case $|\zeta|\approx 1$. For the purposes of the rest of this section, we thus assume that the symbol q (and hence the flow $\Theta_{s,r}$) is homogeneous of degree one in ζ, and agrees with our previous definition of q on the above region (which had smoothly set $q=|\zeta|$ outside the region $|\zeta_3|\approx\mu$, $|\zeta_2|\lesssim\frac12\mu$.)

Theorem 5.1. *Let $z_{s,r}$ and $\zeta_{s,r}$ be defined as functions of (z,ζ) by (5.2). Let d_ζ and d_z respectively denote the ζ-gradient and z-gradient operators. Then, for $\zeta_3=1$ and $\zeta_2\approx\theta$ (respectively $|\zeta_2|\le\mu^{-\frac12}$ in case $\theta=\mu^{-\frac12}$) the following bounds hold.*

$$\begin{aligned} &\big|d_z z_{s,r}-I\big|\lesssim|s-r|\,, && \big|d_\zeta z_{s,r}\big|\lesssim|s-r|\,,\\ &\big|d_\zeta\zeta_{s,r}-I\big|\lesssim|s-r|\,, && \big|d_z\zeta_{s,r}\big|\lesssim 1\,. \end{aligned}\tag{5.5}$$

Also,

$$\begin{aligned} &\big|d_z^2 z_{s,r}\big|\lesssim\langle\mu^{\frac12}|s-r|\,\rangle\,, && \big|d_z^2\zeta_{s,r}\big|\lesssim\mu^{\frac12}\\ &\big|d_zd_\zeta z_{s,r}\big|\lesssim|s-r|\,\langle\mu^{\frac12}|s-r|\,\rangle\,, && \big|d_zd_\zeta\zeta_{s,r}\big|\lesssim\langle\mu^{\frac12}|s-r|\,\rangle\,. \end{aligned}\tag{5.6}$$

Furthermore, for $k\ge 2$,

$$\big|d_\zeta^k z_{s,r}\big| \;+\; \big|d_\zeta^k\zeta_{s,r}\big| \;\lesssim\; |s-r|\,\langle\mu^{\frac12}|s-r|\,\rangle^{k-1}\,. \tag{5.7}$$

Proof. We start with the relation

$$z_{s,r}=z+\int_r^s(d_\zeta q)(t,z_{t,r},\zeta_{t,r})\,dt\,,\qquad \zeta_{s,r}=\zeta-\int_r^s(d_zq)(t,z_{t,r},\zeta_{t,r})\,dt\,.$$

Differentiating with respect to z and ζ yields

$$\begin{pmatrix} dz_{s,r} \\ \\ d\zeta_{s,r} \end{pmatrix} = \begin{pmatrix} dz \\ \\ d\zeta \end{pmatrix} + \int_r^s M(t, z_{t,r}, \zeta_{t,r}) \cdot \begin{pmatrix} dz_{t,r} \\ \\ d\zeta_{t,r} \end{pmatrix} dt\,, \tag{5.8}$$

where

$$M = \begin{pmatrix} \bigl(d_z d_\zeta q\bigr) & \bigl(d_\zeta d_\zeta q\bigr) \\ \\ -\bigl(d_z d_z q\bigr) & -\bigl(d_\zeta d_z q\bigr) \end{pmatrix}$$

The key estimate is that, for $i + j = 2$,

$$\int_r^s \bigl|(d_z^i d_\zeta^j q)(t, z_{t,r}, \zeta_{t,r})\bigr|\, dt \lesssim \begin{cases} |s-r|\,, & i \le 1\,, \\ 1\,, & i = 2\,. \end{cases}$$

This follows by (4.1) (recall that z_2 equals x_2), and the fact that $|(d_t z_{t,r})_2| \approx \theta$ in case $\theta > \mu^{-\frac12}$. In the case $\theta = \mu^{-\frac12}$, estimate (4.1) shows that the integrand is in fact uniformly bounded.

An application of the Gronwall lemma yields

$$|dz_{t,r}| \lesssim 1\,, \qquad |d\zeta_{t,r}| \lesssim 1\,,$$

and plugging this into (5.8) yields (5.5).

To control higher order derivatives we proceed by induction. For $k \ge 2$, we write

$$\begin{pmatrix} d_\zeta^k z_{s,r} \\ \\ d_\zeta^k \zeta_{s,r} \end{pmatrix} = \int_r^s M(t, z_{t,r}, \zeta_{t,r}) \cdot \begin{pmatrix} d_\zeta^k z_{t,r} \\ \\ d_\zeta^k \zeta_{t,r} \end{pmatrix} dt + \int_r^s \begin{pmatrix} E_1(t) \\ \\ E_2(t) \end{pmatrix} dt$$

where $E_1(t)$ is a sum of terms of the form

$$\bigl(d_z^i d_\zeta^{j+1} q\bigr)(t, z_{t,r}, \zeta_{t,r}) \cdot \bigl(d_\zeta^{k_1} z_{t,r}\bigr) \cdots \bigl(d_\zeta^{k_i} z_{t,r}\bigr)\bigl(d_\zeta^{k_{i+1}} \zeta_{t,r}\bigr) \cdots \bigl(d_\zeta^{k_{i+j}} \zeta_{t,r}\bigr)$$

and E_2 is similarly a sum of such terms, but with $d_z^{i+1} d_\zeta^j q$. In both cases, $k_n < k$ for each n, and $k_1 + \cdots + k_{i+j} = k$. By induction we may thus assume that the estimates (5.5) and (5.7) hold for all terms arising in E_1 and E_2.

The bound (4.1) implies, as above, that for $|\zeta| = 1$

$$\int_r^s |(d_z^{i+1} d_\zeta^j q)(t, z_{t,r}, \zeta_{t,r})|\, dt \lesssim \begin{cases} |s-r|\,, & i = 0 \\ \mu^{\frac12(i-1)}\,, & i \ge 1 \end{cases} \tag{5.9}$$

The induction hypothesis yields that

$$\bigl|\bigl(d_\zeta^{k_1} z_{t,r}\bigr) \cdots \bigl(d_\zeta^{k_i} z_{t,r}\bigr)\bigl(d_\zeta^{k_{i+1}} \zeta_{t,r}\bigr) \cdots \bigl(d_\zeta^{k_{i+j}} \zeta_{t,r}\bigr)\bigr| \lesssim |t-r|^i \,\langle \mu^{\frac12} |t-r|\,\rangle^{k-i-j}\,.$$

Together these yield

$$\int_r^s |E_2(t)|\, dt \lesssim |s-r|\,\langle \mu^{\frac12} |s-r|\,\rangle^{k-1}\,,$$

and the same holds for E_1. The estimate (5.7) follows by the Gronwall lemma.

To establish the first line of (5.6), we write

$$\begin{pmatrix} d_z^2 z_{s,r} \\ \\ d_z^2 \zeta_{s,r} \end{pmatrix} = \int_r^s M(t, z_{t,r}, \zeta_{t,r}) \cdot \begin{pmatrix} d_z^2 z_{t,r} \\ \\ d_z^2 \zeta_{t,r} \end{pmatrix} \, dt + \int_r^s \begin{pmatrix} E_1(t) \\ \\ E_2(t) \end{pmatrix} \, dt \,, \tag{5.10}$$

where now

$$\int_r^s |E_1(t)| \, dt \lesssim 1 \,, \qquad \int_r^s |E_2(t)| \, dt \lesssim \mu^{\frac12} \,.$$

A first application of Gronwall yields $|d_z^2 z_{s,r}| + |d_z^2 \zeta_{s,r}| \lesssim \mu^{\frac12}$, and plugging this into (5.10) and using (5.9) yields the first line of (5.6). The second line follows by similar considerations. □

Corollary 5.2. *The following bounds hold for $\zeta_3 = 1$ and $\zeta_2 \approx \theta$,*

$$\left| d_\zeta z_{s,r} - \int_r^s d_\zeta^2 q(t, \Theta_{t,r}(z,\zeta)) \, dt \right| \le c \, |s-r|^2 \,,$$

where c can be made small by taking the constant c_0 in condition (2.3) *small.*

Proof. Given c, choosing the constant c_0 small yields the bounds

$$|d_\zeta d_{z,\zeta} q| \le c \,, \qquad \int_r^s |d_z^2 q(t, \Theta_{t,r}(z,\zeta))| \, dt \le c \,.$$

Together with the bounds (5.5), plugging this into (5.8) yields successively the bounds

$$|d_\zeta \zeta_{s,r} - I| \le c \, |s-r| \,, \qquad \left| d_\zeta z_{s,r} - \int_r^s d_\zeta^2 q(t, \Theta_{t,r}(z,\zeta)) \, dt \right| \le c \, |s-r|^2 \,. \quad \square$$

Lemma 5.3. *Suppose that $|\zeta| \approx \mu$, and $\bar\theta$ is a number with $\bar\theta \ge \mu^{-\frac12}$ and $\mu \bar\theta^2 |s-r| \le 1$. Then, for all α and j,*

$$\left| (\zeta \cdot d_\zeta)^j \, (\mu \bar\theta \partial_\zeta)^\alpha \mu^{\frac32} \bar\theta \, d_\zeta z_{s,r} \right| \lesssim 1 \,, \tag{5.11}$$

and for all α and j with $j + |\alpha| \ge 1$,

$$\left| (\zeta \cdot d_\zeta)^j \, (\mu \bar\theta \partial_\zeta)^\alpha \mu \bar\theta \langle d_\zeta \zeta_{s,r}, y - z_{s,r} \rangle \right| \lesssim \langle \mu^{\frac12} |y - z_{s,r}| \, \rangle \,. \tag{5.12}$$

Proof. First consider (5.11). By homogeneity of $z_{s,r}$ and its derivatives, it suffices to consider $j=0$. We then have, by (5.5)–(5.7) and homogeneity,

$$\left| (\mu \bar\theta \partial_\zeta)^\alpha \mu^{\frac32} \bar\theta \, d_\zeta z_{s,r} \right| \lesssim \mu^{\frac12} \bar\theta^{|\alpha|+1} |s-r| \, \langle \mu^{\frac12} |s-r| \, \rangle^{|\alpha|} \le \langle \mu^{\frac12} \bar\theta \, |s-r| \, \rangle^{|\alpha|+1} \lesssim 1 \,.$$

Here we use that $\mu \bar\theta^2 \ge 1$, so that $\mu^{\frac12} \bar\theta |s-r| \le 1$. For (5.12), note that if $|\alpha| = 0$ and $j \ne 0$ then the term vanishes by homogeneity, so we may assume $|\alpha| \ge 1$. By homogeneity we may also restrict to the case $j = 0$. First consider the case where all derivatives fall on $\zeta_{s,r}$. The resulting term is bounded by

$$\mu \bar\theta^{|\alpha|+1} |s-r| \, \langle \mu^{\frac12} |s-r| \, \rangle^{|\alpha|} \, |y - z_{s,r}| \lesssim \langle \mu^{\frac12} \bar\theta \, |s-r| \, \rangle^{|\alpha|} \langle \mu^{\frac12} |y - z_{s,r}| \, \rangle \lesssim \langle \mu^{\frac12} |y - z_{s,r}| \, \rangle \,.$$

If one or more derivatives falls on $z_{s,r}$, the term is bounded by

$$\mu \bar\theta^{|\alpha|+1} |s-r| \, \langle \mu^{\frac12} |s-r| \, \rangle^{|\alpha|-1} \lesssim \mu \bar\theta^2 |s-r| \, \langle \mu^{\frac12} \bar\theta \, |s-r| \, \rangle^{|\alpha|-1} \lesssim 1 \,.$$

□

Recall that the kernel we are proving (5.4) for is

$$K(r,x';s,y') = \mu \int e^{i\langle \zeta, x'-z\rangle - i\langle \zeta_{s,r}, y'-z_{s,r}\rangle} g\big(\mu^{\frac12}(x'-z)\big)\, g\big(\mu^{\frac12}(y'-z_{s,r})\big)\,\beta_\theta(\zeta)\,dz\,d\zeta\,,$$

where $\beta_\theta(\zeta)$ is a cutoff to $\zeta_3 \approx \mu$ and $\zeta_2 \approx \theta\mu$, respectively $|\zeta_2| \lesssim \mu^{\frac12}$ in case $\theta = \mu^{-\frac12}$.

In what follows, for the case $\theta > \mu^{-\frac12}$ we will need to consider finer angular decompositions in ζ, depending on $|s-r|$. We will assume, for the following theorem, that $\beta_{\bar\theta}(\zeta)$ is a smooth cutoff to a set of the form

$$\zeta_3 \approx \mu\,, \qquad \zeta_2 \approx \theta\mu\,, \qquad \Big|\,\frac{\zeta_2}{\zeta_3} - \theta'\,\Big| \lesssim \bar\theta\,,$$

where $\theta' \approx \theta$, and where $\mu^{-\frac12} \le \bar\theta \le \theta$. For $\theta = \mu^{-\frac12}$, we need consider only $\beta_{\bar\theta} = \beta_\theta$.

Theorem 5.4. *Consider the kernel K with $\beta_\theta(\zeta)$ replaced by $\beta_{\bar\theta}(\zeta)$, with $\beta_{\bar\theta}$ as above. Suppose that $\mu\bar\theta^2|s-r| \le 1$. Fix a vector ξ' in the support of $\beta_{\bar\theta}(\zeta)$, and let $(x'_{s,r},\nu_{s,r})$ be the projection of $\Theta_{s,r}(x',\xi')$ onto the cosphere bundle. Thus, $x'_{s,r} = z_{s,r}$ and $\nu_{s,r} = |\zeta_{s,r}|^{-1}\zeta_{s,r}$ if $z = x'$ and $\zeta = \xi'$. Then*

$$|K(r,x';s,y')| \lesssim \mu^2\bar\theta\,\big(\,1 + \mu\bar\theta\,|\,y' - x'_{s,r}| + \mu\,|\,\langle \nu_{s,r}, y' - x'_{s,r}\rangle\,|\,\big)^{-N}\,.$$

Proof. We introduce the differential operators, where $z_{s,r}$ and $\zeta_{s,r}$ are as in (5.2),

$$L_1 = \frac{1 - i\big(\,\langle \zeta, x'-z\rangle - \langle \zeta_{s,r}, y'-z_{s,r}\rangle\,\big)\,\langle \zeta, d_\zeta\rangle}{1 + \big|\,\langle \zeta, x'-z\rangle - \langle \zeta_{s,r}, y'-z_{s,r}\rangle\,\big|^2}\,,$$

and

$$L_2 = \frac{1 - i\mu\bar\theta\,\big(\,x' - z - d_\zeta\zeta_{s,r}\cdot(y'-z_{s,r})\,\big)\cdot d_\zeta}{1 + \mu^2\bar\theta^2\,\big|\,x' - z - d_\zeta\zeta_{s,r}\cdot(y'-z_{s,r})\big|^2}\,.$$

Each of these preserves the phase function in K, and an integration by parts argument, using the estimates (5.11) and (5.12), bounds $|K(r,x';s,y')|$ by the following integral

$$\mu\int \big(\,1 + \mu\bar\theta\,\big|\,x' - z - d_\zeta\zeta_{s,r}\cdot(y'-z_{s,r})\big|\,\big)^{-N}\big(\,1 + \big|\,\langle \zeta, x'-z\rangle - \langle \zeta_{s,r}, y'-z_{s,r}\rangle\,\big|\,\big)^{-N}$$
$$\times\,\big(\,1 + \mu^{\frac12}|x'-z|\,\big)^{-N}\big(\,1 + \mu^{\frac12}|y'-z_{s,r}|\,\big)^{-N}\,dz\,d\zeta\,,$$

where the integral is over the support of $\beta_{\bar\theta}(\zeta)$, which has volume $\mu^2\bar\theta$. We will show below that

$$\mu\bar\theta\,|\,d_\zeta\zeta_{s,r}\cdot(x'_{s,r}-z_{s,r}) - (x'-z)| + |\langle \zeta_{s,r}, x'_{s,r}-z_{s,r}\rangle - \langle \zeta, x'-z\rangle| \lesssim 1 + \mu\,|x'-z|^2\,. \tag{5.13}$$

This implies that the integrand is dominated by

$$\big(\,1 + \mu\bar\theta\,|d_\zeta\zeta_{s,r}\cdot(y'-x'_{s,r})| + |\langle \zeta_{s,r}, y'-x'_{s,r}\rangle|\,\big)^{-N}\big(\,1 + \mu^{\frac12}|x'-z|\,\big)^{-N}\,.$$

By (5.5), the matrix $d_\zeta\zeta_{s,r}$ is invertible. Also by (5.5), the angle of $\zeta_{s,r}$ to $\mu\nu_{s,r}$ is less than $\bar\theta + |x'-z|$. Since $\mu\bar\theta \ge \mu^{\frac12}$, and $|\zeta_{s,r}| \approx \mu$, together these dominate the integrand by

$$\big(\,1 + \mu\bar\theta\,|\,y' - x'_{s,r}| + \mu\,|\langle \nu_{s,r}, y'-x'_{s,r}\rangle|\,\big)^{-N}\big(\,1 + \mu^{\frac12}|x'-z|\,\big)^{-N}\,,$$

from which the theorem follows easily.

We now establish (5.13). Consider the first term on the left. By homogeneity, we may assume that $1=|\zeta|=|\xi'|$, so that $|\zeta-\xi'|\le\bar\theta$. By (5.6) and Taylor's theorem, we then have

$$|x'_{s,r}-z_{s,r}-(d_z z_{s,r})(x'-z)-(d_\zeta z_{s,r})(\xi'-\zeta)| \\ \lesssim \langle\mu^{\frac12}|s-r|\,\rangle\,|x'-z|^2+|s-r|\,\langle\mu^{\frac12}|s-r|\,\rangle\big(\bar\theta^2+\bar\theta|x'-z|\,\big)\,.$$

After multiplication by $\mu\bar\theta$, each term on the right is bounded by $1+\mu\,|x'-z|^2$. Also,

$$\mu\bar\theta\,|\,(d_\zeta z_{s,r})(\zeta-\xi')|\lesssim\mu\bar\theta^2\,|s-r|\le 1\,.$$

Since $d\zeta_{s,r}\wedge dz_{s,r}=d\zeta\wedge dz$, we have

$$\partial_{\zeta_i}\zeta_{s,r}\cdot\partial_{z_j}z_{s,r}-\partial_{\zeta_i}z_{s,r}\cdot\partial_{z_j}\zeta_{s,r}=\delta_{ij}\,,$$

where $\cdot$ pairs the $z_{s,r}$ and $\zeta_{s,r}$ indices. By (5.5), we have

$$\mu\bar\theta\,\big|d_\zeta z_{s,r}\big|\,\big|d_z\zeta_{s,r}\big|\,|x'-z|\lesssim\mu^{\frac12}|x'-z|\,.$$

Together, this yields

$$\mu\bar\theta\,|\,d_\zeta\zeta_{s,r}\cdot(x'_{s,r}-z_{s,r})-(x'-z)|\lesssim 1+\mu\,|x'-z|^2\,,$$

which concludes the bound for the first term.

To handle the second term, it suffices by homogeneity to show that, for $|\zeta|=|\xi'|=1$,

$$|\langle\zeta_{s,r},x'_{s,r}-z_{s,r}\rangle-\langle\zeta,x'-z\rangle|\lesssim|x'-z|^2+\bar\theta^2|s-r|\,.$$

We calculate

$$\frac{d}{ds}\langle\zeta_{s,r},x'_{s,r}-z_{s,r}\rangle= \\ -\big\langle(d_zq)(s,\Theta_{s,r}(z,\zeta)),x'_{s,r}-z_{s,r}\big\rangle+\big\langle\zeta_{s,r},(d_\zeta q)(s,\Theta_{s,r}(x',\xi'))-(d_\zeta q)(s,\Theta_{s,r}(z,\zeta))\big\rangle\,.$$

By homogeneity, the right hand side equals

$$q(s,\Theta_{s,r}(x',\xi'))-q(s,\Theta_{s,r}(z,\zeta))-\big(\Theta_{s,r}(x',\zeta)-\Theta_{s,r}(z,\zeta)\big)\cdot(d_{z,\zeta}q)(s,\Theta_{s,r}(z,\zeta)) \tag{5.14}$$

plus an error which, since q_ζ is Lipschitz, is bounded by

$$|\Theta_{s,r}(x',\xi')-\Theta_{s,r}(z,\zeta)|^2\lesssim|x'-z|^2+\bar\theta^2\,. \tag{5.15}$$

Let $\gamma_\sigma(t)=\sigma\Theta_{s,r}(x',\xi')+(1-\sigma)\Theta_{s,r}(z,\zeta)$. Then (5.14) equals

$$\int_0^1\int_r^s(1-\sigma)\big(\Theta_{s,r}(x',\xi')-\Theta_{s,r}(z,\zeta)\big)^2(d^2_{z,\zeta}q)(t,\gamma_\sigma(t))\,dt\,d\sigma\,.$$

By (5.15), the integral of terms involving $d_zd_\zeta q$ and $d^2_\zeta q$ are bounded by

$$|s-r|\,|x'-z|^2+|s-r|\,\bar\theta^2\le|x'-z|^2+\bar\theta^2|s-r|\,.$$

The integral of terms involving d^2_zq are bounded by

$$\Big(\sup_{r\le t\le s}|x'_{t,r}-z_{t,r}|^2\Big)\,\sup_\sigma\int_r^s|(d^2_zq)(t,\gamma_\sigma(t))|\,dt\lesssim|x'-z|^2+\bar\theta^2|s-r|^2\,,$$

where we use (5.5), (4.1), and the fact that $(\dot\gamma_\sigma)_2\approx\theta$ in the case $\theta>\mu^{-\frac12}$. $\square$

Proof of estimate (5.4)**.** We establish (5.4) by showing that

$$\sup_{x_2,x_3,y_2} \int |K(r,x';s,y')|\,dy_3 \lesssim \mu\theta\,\bigl(\,1+\mu\theta^2|s-r|\,\bigr)^{-\frac12}\,. \tag{5.16}$$

Transposing (s,y') and (r,x') in the formula for K leads to the same kernel if $\beta_\theta(\zeta)$ is replaced by $\beta_\theta(\zeta_{r,s}(y',\zeta))$, and the same proof will show that

$$\sup_{x_2,y_2,y_3} \int |K(r,x';s,y')|\,dx_3 \lesssim \mu\theta\,\bigl(\,1+\mu\theta^2|s-r|\,\bigr)^{-\frac12}\,,$$

yielding (5.4) by Schur's lemma.

Suppose first that $\mu\theta^2|s-r|\le 1$. Then (5.16) follows immediately from Theorem 5.4 with $\bar\theta=\theta$, since $\nu_{s,r}=|\zeta_{s,r}|^{-1}\zeta_{s,r}$ is within a small angle of the ξ_1 axis.

If $\mu\theta^2|s-r|>1$, we let $\bar\theta=\mu^{-\frac12}|s-r|^{-\frac12}$, and decompose K into a sum of terms by writing $\beta_\theta(\zeta)=\sum_j\beta_j(\zeta)$, with each $\beta_j(\zeta)$ a cutoff to a sector of angle $\bar\theta$.

We fix η^j in the support of $\beta_j(\zeta)$, with

$$(\eta^j)_3=\mu\,,\quad \text{and}\quad |(\eta^i)_2-(\eta^j)_2|\approx \mu\,\bar\theta\,|i-j|\,.$$

We then have decomposed $K=\sum_j K_j$, where by Theorem (5.4)

$$|K_j(r,x';s,y')|\lesssim \mu^2\bar\theta\,\bigl(\,1+\mu\bar\theta\,|\,y'-w^j_{s,r}|+\mu\,|\langle\nu^j_{s,r},y'-w^j_{s,r}\rangle|\,\bigr)^{-N}\,,$$

where $(w^j_{s,r},\nu^j_{s,r})$ is the projection onto the cosphere bundle of $\Theta_{s,r}(x',\eta^j)$. Since $(\nu^j_{s,r})_3\approx 1$, we have

$$\int |K_j(r,x';s,y')|\,dy_3\lesssim \mu\bar\theta\,\bigl(1+\mu\bar\theta\,|\,y_2-(w^j_{s,r})_2|\,\bigr)^{-N}\,.$$

Since $\mu\bar\theta\approx\mu\theta\,(1+\mu\theta^2|s-r|\,)^{-\frac12}$, it suffices to show that

$$\sup_{x_2,x_3,y_2}\sum_j\bigl(1+\mu\bar\theta\,|\,y_2-(w^j_{s,r})_2|\,\bigr)^{-N}\lesssim 1\,,$$

which we do by recalling that $\mu\bar\theta^2|s-r|=1$, and showing that

$$|(w^i_{s,r})_2-(w^j_{s,r})_2|\approx\bar\theta\,|s-r|\,|i-j|\,.$$

We finally show this by noting that, for $\zeta_3=1$ and $|\zeta_2|\le\frac12$, we have $d^2_{\zeta_2}q\approx 1$. Corollary 5.2 thus yields $d_{\zeta_2}(z_{s,r})_2\approx s-r$ for such ζ. Consequently,

$$|z_{s,r}(x',\eta^i)-z_{s,r}(x',\eta^j)|\approx\mu^{-1}\,|s-r|\,|\,(\eta^i)_2-(\eta^j)_2|\approx\bar\theta\,|s-r|\,|i-j|\,.\quad\square$$

6. Energy Flux Estimates

In this section we complete the proof of Theorem 2.2 by establishing the endpoint estimates where $q=8$. We do this by establishing the nested square-summability condition (3.1). Recall that we are assuming

$$D_1u_\lambda-P_\lambda u_\lambda=F_\lambda\,,$$

where $2P_\lambda=p_\lambda(x,D')+p_\lambda(x,D')^*$, and we write

$$D_1u_j-P_ju_j=F_j+G_j\,,$$

where $u_j = \beta_j(D')u_\lambda$, the operator $P_j = p_j(x, D')$ has symbol truncated to x'- frequencies less than $\lambda^{\frac12}\theta_j^{-\frac12}$, and

$$F_j = \beta_j(D')F_\lambda + [\beta_j(D'), P_j]u_\lambda + \beta_j(D')\big(P_\lambda - p_\lambda(x, D')\big)u_\lambda\,, \tag{6.1}$$

$$G_j = \beta_j(D')\big(p_\lambda(x, D') - p_j(x, D')\big)u_\lambda\,. \tag{6.2}$$

Let

$$c_{j,k} = \|u_j\|_{L^\infty L^2(S_{j,k})} + \lambda^{\frac14}\theta_j^{\frac14}\,\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^{-1}u_j\|_{L^2(S_{j,k})} + \|F_j\|_{L^1L^2(S_{j,k})} + \lambda^{-\frac14}\theta_j^{-\frac14}\,\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^2 G_j\|_{L^2(S_{j,k})}\,. \tag{6.3}$$

We need to show that

$$\sum_{j=1}^{N_\lambda} c_{j,k(j)}^2 \lesssim \|u_\lambda\|^2_{L^\infty L^2(S)} + \|F_\lambda\|^2_{L^2(S)}\,, \tag{6.4}$$

where $k(j)$ denotes any sequence of values for k such that the slabs $S_{j,k(j)}$ are nested, in that for $j \ge 1$ we have $S_{j+1,k(j+1)} \subset S_{j,k(j)}$. The analogous bound for $j < 0$ will follow by an identical proof.

6.1. **Estimates on u_j.** We begin by establishing the square-summability estimates for the first two terms on the right hand side of (6.3). By translation invariance we may assume each $S_{j,k(j)}$ contains $x_1 = 0$. We then take S_j to be the slab $[0, \varepsilon\, 2^{-j}] \times \mathbb{R}^2$, and will show that

$$\sum_i \Big(\|u_i\|^2_{L^\infty L^2(S_i)} + \lambda^{\frac12}\theta_i^{\frac12}\,\|\langle\lambda^{\frac12}\theta_i^{-\frac12}x_2\rangle^{-1}u_i\|^2_{L^2(S_i)}\Big) \lesssim \|u_\lambda\|^2_{L^\infty L^2} + \|F_\lambda\|^2_{L^2}\,.$$

The same bounds will hold for $x_1 \in [-\varepsilon\, 2^{-j}, 0]$.

Since $F_\lambda \in L^1_{x_1}L^2_{x'}$, by Duhamel we can reduce matters to the homogeneous case $F_\lambda = 0$. Assume this, and let $f(x') = u_\lambda(0, x')$. Let W denote the solution operator for the Cauchy problem associated to P_λ, so that $u_\lambda = Wf$, It then suffices to show that

$$\|\beta_i(D')W\beta_j(D')f\|_{L^\infty L^2(S_i)} + \lambda^{\frac14}\theta_i^{\frac14}\,\|\langle\lambda^{\frac12}\theta_i^{-\frac12}x_2\rangle^{-1}\beta_i(D')W\beta_j(D')f\|_{L^2(S_i)} \lesssim 2^{-\frac34|i-j|}\,\|f\|_{L^2}\,. \tag{6.5}$$

To prove (6.5), we will construct for each given j a function v which satisfies the following conditions.

$$v(0, x') = \beta_j(D')f(x')\,, \qquad \beta_i(D)v = 0 \quad \text{if} \quad |i - j| \ge 5\,, \tag{6.6}$$

$$\|v\|_{L^\infty L^2(S_j)} \lesssim \|f\|_{L^2} \tag{6.7}$$

$$\lambda^{\frac14}\theta_j^{\frac14}\,\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^{-1}v\|_{L^2(S_j)} \lesssim \|f\|_{L^2}\,, \tag{6.8}$$

and such that

$$\|D_1v - P_\lambda v\|_{L^1L^2(S_j)} \lesssim (\lambda^{\frac12}\theta_j^{\frac32})^{-\frac12}\,\|f\|_{L^2}\,, \tag{6.9}$$

$$\|D_1v - P_\lambda v\|_{L^1L^2(S_j)} \lesssim (\lambda^{\frac12}\theta_j^{\frac32})^{-1}\,\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle f\|_{L^2}\,. \tag{6.10}$$

Let us show that these imply the estimate (6.5). Consider the first term on the left hand side of (6.5). We will prove the stronger statement

$$\|\beta_i(D')W(r)\beta_j(D')f\|_{L^2_{x'}} \lesssim 2^{-\frac{3}{4}|i-j|}\,\|f\|_{L^2}\,, \quad |r|\le \varepsilon\,\max(2^{-i},2^{-j})\,. \tag{6.11}$$

By self adjointness (the adjoint of $W(r)$ is the wave map going the other way), we can then assume that $\theta_j = 2^{-j} \ge \theta_i = 2^{-i}$. This assumption now means we need to control data at angle 2^{-j} for time $\varepsilon\,2^{-j}$.

We write $W\beta_j(D')f = v - w$. The desired estimate holds for the v term by (6.7), since we may assume $|i-j|\le 4$ by (6.6) (and we may shrink ε by a factor of 16.)

To control w, we note that

$$w(0,x') = 0\,, \qquad D_1 w - P_\lambda w = D_1 v - P_\lambda v\,.$$

Energy estimates and (6.9) thus yield

$$\|w\|_{L^\infty L^2(S_j)} \le (\lambda^{\frac12}\theta_j^{\frac32})^{-\frac12}\,\|f\|_{L^2}\,. \tag{6.12}$$

Since $\lambda^{\frac12}\ge 2^{\frac32 i}$, this yields the desired bound on u.

To estimate the second term in (6.5), we first consider the case $\theta_i\le\theta_j$. We again write $W\beta_j(D')f = v-w$, and note that the desired estimate on v follows by (6.8) and (6.6). (The operator $\beta_i(D')$ preserves the L^2-weight $\langle\lambda^{\frac12}\theta_i^{-\frac12}x_2\rangle^{-1}$ since $\lambda^{\frac12}\theta_i^{-\frac12}\le 2^{-i}\lambda$.)

The estimate on w for $\theta_i\le\theta_j$ follows by (6.12),

$$\lambda^{\frac14}\theta_i^{\frac14}\|w\|_{L^2(S_i)} \le \lambda^{\frac14}\theta_i^{\frac34}\|w\|_{L^\infty L^2(S_j)} \lesssim \theta_j^{-\frac34}\theta_i^{\frac34}\|f\|_{L^2}\,.$$

Now consider the case $\theta_j\le\theta_i$. The above steps handle the case $|i-j|\le 4$, so we assume $i\ge j+5$. We take adjoints to reduce matters to showing that, for $j\ge i+5$,

$$\Big\|\int_{|s|\le\varepsilon\theta_i}\beta_j(D')W(s)^*\beta_i(D')F(s,\,\cdot\,)\,ds\Big\|_{L^2} \lesssim \lambda^{-\frac14}\theta_i^{-\frac14}\,2^{-\frac34|i-j|}\|\langle\lambda^{\frac12}\theta_i^{-\frac12}x_2\rangle F\|_{L^2(S_i)}\,.$$

This bound, in turn, follows from showing that, for $|r|\le\varepsilon\,2^{-i}$ and $j\ge i+5$,

$$\|\beta_j(D')W^*(r)\beta_i(D')f\|_{L^2} \lesssim (\lambda^{\frac12}\theta_i^{\frac32})^{-1}\|\langle\lambda^{\frac12}\theta_i^{-\frac12}x_2\rangle f\|_{L^2}\,.$$

We may replace $W^*(r)$ by $W(r)$, since W^* is the Cauchy map for data at $x_1=r$ to $x_1=0$, and after exchanging i and j this bound is a consequence of (6.10).

6.2. **The construction of v.** We assume that θ_j is now fixed, and rescale spatial variables by θ_j. We thus need to construct v on the slab $S=[0,\varepsilon]\times\mathbb{R}^2$. As before, let $\mu=\lambda\theta_j$, and let $\beta_j(D')$ denote the rescaled localization operators, which will localize to $\xi_2\approx\theta_j\mu\,,\ \xi_3\approx\mu$. Let f denote the rescaled initial data $\beta_j(D')f(\theta_j\,\cdot\,)$.

In these rescaled variables it suffices to produce v satisfying

$$v(0,x') = f(x')\,, \qquad \beta_i(D)v = 0 \quad \text{if} \quad |i-j|\ge 5\,, \tag{6.13}$$

$$\|v\|_{L^\infty L^2(S)} \lesssim \|f\|_{L^2} \tag{6.14}$$

$$\mu^{\frac14}\theta_j^{\frac12}\|\langle\mu^{\frac12}x_2\rangle^{-1}v\|_{L^2(S)} \lesssim \|f\|_{L^2}\,, \tag{6.15}$$

and such that

$$\|D_1 v - Q_\mu v\|_{L^2(S)} \lesssim (\mu^{\frac12}\theta_j)^{-\frac12}\,\|f\|_{L^2}\,, \tag{6.16}$$

$$\|D_1 v - Q_\mu v\|_{L^1L^2(S)} \lesssim (\mu^{\frac12}\theta_j)^{-1}\,\|\langle \mu^{\frac12}x_2\rangle f\|_{L^2}\,. \tag{6.17}$$

Here Q_μ is the rescaled operator P_λ, which has symbol truncated to x'-frequencies less than $c\mu$.

We will construct v using the modified FBI/Cordoba-Fefferman transform T_μ introduced in §4. The key idea is that this transform conjugates the operator Q_μ to the Hamiltonian flow field, plus a bounded error which is roughly local. Precisely, we will show that

$$T_\mu Q_\mu T_\mu^* = D_q + K,$$

where D_q is the Hamiltonian vector field of the symbol q (which we recall is frequency localized to $\mu^{\frac12}$), and where K is an operator on $L^2_{x',\xi'}$, depending on parameter x_1, for which we establish weighted L^2 estimates.

The transform $\tilde u = T_\mu u$ of the exact solution u to $D_1u - Q_\mu u = 0$, with initial data f, satisfies

$$D_1\tilde u - D_q\tilde u = K\tilde u\,, \qquad \tilde u(0,x',\xi') = \tilde f(x',\xi')\,.$$

The operator K will introduce terms which are well-behaved after integration along the flow of $D_1 - D_q$ at angle θ_j. We will construct the approximate solution v by truncating the operator K to such angles. For this purpose we introduce cutoffs $\phi_j(\xi')$ and $\psi_j(\xi')$, with slightly larger supports than $\beta_j(\xi')$, such that

$$\operatorname{dist}\bigl(\operatorname{supp}(1-\phi_j),\operatorname{supp}(\beta_j)\bigr) \ge 2^{-j-10}\mu\,,$$
$$\operatorname{dist}\bigl(\operatorname{supp}(1-\psi_j),\operatorname{supp}(\phi_j)\bigr) \ge 2^{-j-10}\mu\,,$$

and also that

$$\operatorname{dist}\bigl(\operatorname{supp}(\psi_j),\operatorname{supp}(\beta_i)\bigr) \ge 2^{-j-10}\mu \quad \text{if} \quad |i-j|\ge 5\,.$$

The ξ'-support of $\tilde f$ lies in the $c\mu^{-\frac12}$ neighborhood of the support of $\beta_j(\xi')$. Since $c \ll 1$, $\theta_j \ge \mu^{-\frac12}$, and $|d_x q(x,\xi')| \le c\,\theta_j\,|\xi'|$, we can assume that every integral curve of $D_1 - D_q$ passing through this neighborhood remains ξ'-distance at least $2^{-10}\mu\theta_j$ away from the support of $(1-\phi_j)$.

Furthermore, we can assume that any integral curve of $D_1 - D_q$ passing at any point through the support of ψ_j does not meet the $c\mu^{-\frac12}$ neighborhood of the support of $\beta_i(\xi')$, provided $|i-j|\ge 5$.

We will take $v = T_\mu^*\tilde v$ where $\tilde v$ solves

$$D_1\tilde v - D_q\tilde v = \psi_j\,K\tilde v\,, \qquad \tilde v(0,x',\xi') = \tilde f(x',\xi')\,. \tag{6.18}$$

The cutoff ψ_j restricts the right hand side to $\xi_2 \approx \theta_j\mu$, where the integral of K along $D_1 - D_q$ is under control. Furthermore, since the support of $\tilde v$ will be contained in the union of the integral curves of $D_1 - D_q$ passing through the support of ψ_j at some point x_1, then v will satisfy $\beta_i(D')v = 0$ for $|i-j|\ge 5$.

Next, since $Q_\mu T_\mu^* = T_\mu^* D_q + T_\mu^* K$, it holds that

$$D_1 v - Q_\mu v = -T_\mu^*\bigl((1-\psi_j)K\tilde v\bigr)\,,$$

so estimates (6.16) and (6.17) will follow from

$$\|(1-\psi_j)\,K\tilde{v}\|_{L^2(\tilde{S})} \lesssim (\mu^{\frac12}\theta_j)^{-\frac12}\,\|\tilde{f}\|_{L^2}\,, \tag{6.19}$$

and

$$\|(1-\psi_j)\,K\tilde{v}\|_{L^1L^2(\tilde{S})} \lesssim (\mu^{\frac12}\theta_j)^{-1}\|\langle\mu^{\frac12}x_2\rangle\tilde{f}\|_{L^2}\,. \tag{6.20}$$

where $\tilde{S}=[0,\varepsilon]\times\mathbb{R}^4_{x',\xi'}$.

We thus need to show that $K\tilde{v}$ is small away from the set $\xi_2\approx\theta_j\mu$, which we do by establishing weighted norm estimates on $\tilde{v}$, and decay estimates on the kernel K. The weights involve the natural distance function on $\mathbb{R}^4_{x',\xi'}$ associated to the Cordoba-Fefferman transform,

$$\mathrm{dist}_\mu(x',\xi';y',\eta')=\mu^{\frac12}\,|x'-y'|+\mu^{-\frac12}|\xi'-\eta'|\,.$$

Let $K(x',\xi';y',\eta')$ denote the integral kernel of K (we supress the parameter x_1). Then we will show that

$$\begin{aligned}|K(x',\xi';y',\eta')| &\lesssim \big(1+\mathrm{dist}_\mu(x',\xi';y',\eta')\big)^{-N}\\ &\quad + c_0\,\mu^{\frac12}\theta_j\langle\,\mu^{\frac12}x_2\rangle^{-N}\langle\,\mu^{-\frac12}|\xi_2-\eta_2|\,\rangle^{-2}\big(1+\mu^{\frac12}|x'-y'|+\mu^{-\frac12}|\xi_3-\eta_3|\,\big)^{-N}\,,\end{aligned} \tag{6.21}$$

where c_0 is the small constant of (2.3).

Let E_0 be the subset of $R^4_{x',\xi'}$

$$E_0=\mathbb{R}^2_{x'}\times\mathrm{supp}(\beta_j(\xi'))\,,$$

and let E_{x_1} be the image of E_0 under the flow along D_1-D_q for time x_1. We consider the weight function

$$M(x,\xi')=M_{x_1}(x',\xi')=1+\mathrm{dist}_\mu(x',\xi';E_{x_1})\,.$$

The weighted norm estimates we establish for solutions of (6.18) are

$$\|M\tilde{v}\|_{L^\infty L^2(\tilde{S})}\lesssim\|M\tilde{f}\|_{L^2}\,, \tag{6.22}$$

$$\|\langle\mu^{\frac12}x_2\rangle^{-1}M\tilde{v}\|_{L^2(\tilde{S})}\lesssim(\mu^{\frac12}\theta_j)^{-\frac12}\|M\tilde{f}\|_{L^2}\,, \tag{6.23}$$

and

$$\|\langle\mu^{\frac12}x_2\rangle^{-2}M\tilde{v}\|_{L^1L^2(\tilde{S})}\lesssim(\mu^{\frac12}\theta_j)^{-1}\|\langle\mu^{\frac12}x_2\rangle M\tilde{f}\|_{L^2}\,. \tag{6.24}$$

Let us show how (6.14)–(6.17) follow from (6.21) and (6.22)–(6.24). The bounds (6.14) and (6.15) are direct consequences of (6.22) and (6.23), since $M=1$ on the support of $\tilde{f}$. Also, (6.16)–(6.17) follow from (6.19)–(6.20), so we focus on (6.19)–(6.20).

We write $K=K_1+K_2$, where the kernels K_1 and K_2 are respectively dominated by the first and second terms on the right hand side of (6.21).

First note that, since $\mathrm{dist}_{\xi'}(\mathrm{supp}(1-\phi_j), E_{x_1}) \geq 2^{-10}\mu\theta_j$ for all x_1, it follows from (6.22)–(6.24) that

$$\|(1-\phi_j)\tilde v\|_{L^\infty L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-1}\|\tilde f\|_{L^2}\,,$$
$$\|\langle\mu^{\frac12}x_2\rangle^{-1}(1-\phi_j)\tilde v\|_{L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-\frac32}\|\tilde f\|_{L^2}\,,$$
$$\|\langle\mu^{\frac12}x_2\rangle^{-2}(1-\phi_j)\tilde v\|_{L^1 L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-2}\|\langle\mu^{\frac12}x_2\rangle\tilde f\|_{L^2}\,.$$

By the bounds on K_1 and K_2 and Schur's Lemma we thus have

$$\|K_1(1-\phi_j)\tilde v\|_{L^\infty L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-1}\|\tilde f\|_{L^2}$$
$$\|K_2(1-\phi_j)\tilde v\|_{L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-\frac12}\|\tilde f\|_{L^2}\,,$$
$$\|K_2(1-\phi_j)\tilde v\|_{L^1 L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-1}\|\langle\mu^{\frac12}x_2\rangle\tilde f\|_{L^2}\,.$$

Next, we note that the integral of K_1, as well as the integral of $(\mu^{\frac12}\theta_j)^{-1}\langle\mu^{\frac12}y_2\rangle K_2$, over the set $|\xi'-\eta'|\geq 2^{-10}\mu\theta_j$ is bounded by $(\mu^{\frac12}\theta_j)^{-1}$, which yields by (6.22)–(6.24) that

$$\|(1-\psi_j)K_1\phi_j\tilde v\|_{L^\infty L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-1}\|\tilde f\|_{L^2}$$
$$\|(1-\psi_j)K_2\phi_j\tilde v\|_{L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-\frac12}\|\tilde f\|_{L^2}\,,$$
$$\|(1-\psi_j)K_2\phi_j\tilde v\|_{L^1 L^2(\tilde S)} \lesssim (\mu^{\frac12}\theta_j)^{-1}\|\langle\mu^{\frac12}x_2\rangle\tilde f\|_{L^2}\,.$$

Together these yield the estimates (6.19) and (6.20).

We turn to the proof of estimates (6.22)–(6.24).

Lemma 6.1. *Take $E\subset\mathbb{R}^4$ and let $M(x',\xi') = 1 + dist_\mu(x',\xi';E)$. Also, let $r_- = \frac12(|r|-r)$. Then, for postive integers k and n, and real number r,*

$$\|M\langle\mu^{\frac12}x_2\rangle^k\langle\mu^{\frac12}(x_2-r)_-\rangle^n K_1 g\|_{L^2(\mathbb{R}^4)} \lesssim \|M\langle\mu^{\frac12}x_2\rangle^k\langle\mu^{\frac12}(x_2-r)_-\rangle^n g\|_{L^2(\mathbb{R}^4)}\,,$$

and

$$\|M\langle\mu^{\frac12}x_2\rangle^k\langle\mu^{\frac12}(x_2-r)_-\rangle^n K_2 g\|_{L^2(\mathbb{R}^4)}$$
$$\lesssim c\,\mu^{\frac12}\theta_j\|M\langle\mu^{\frac12}x_2\rangle^{k-N}\langle\mu^{\frac12}(x_2-r)_-\rangle^n g\|_{L^2(\mathbb{R}^4)}\,.$$

The bounds are uniform over all subsets $E\subset\mathbb{R}^4$ and real numbers r.

Proof. Let K_0 denote the integral kernel

$$K_0(x',\xi';y',\eta') = \bigl(1+\mu^{-\frac12}|\eta_2-\xi_2|\bigr)^{-2}\bigl(1+\mu^{\frac12}|y'-x'|+\mu^{-\frac12}|\eta_3-\xi_3|\bigr)^{-N}\,.$$

By the rapid decrease of K in x' and ξ_3, both estimates are a simple of the following bound

$$\|MK_0 g\|_{L^2} \lesssim \|Mg\|_{L^2}\,.$$

By making the measure preserving change of variables $(x',\xi')\to(\mu^{\frac12}x',\mu^{-\frac12}\xi')$, we may assume $\mu=1$. By the rapid decrease of K_0 in the x' and ξ_3 variables, we may bound

$$\|\,M(x',\xi')\int K_0(x',\xi';y',\eta')\,g(y',\eta')\,dy'\,d\eta'\,\|_{L^2(dx'\,d\xi')}$$
$$\lesssim \|\int M(y',\xi_2,\eta_3)\langle\xi_2-\eta_2\rangle^{-2}\,g(y',\eta')\,d\eta_2\,\|_{L^2(d\xi_2\,dy'\,d\eta_3)}\,.$$

We are thus reduced to the following consequence of the Calderon commutator theorem [1].

Lemma 6.2. *Let $M(r)$ denote a weight function on the real line, satisfying*

$$M(r)\geq 1\,, \qquad |M(r)-M(s)|\leq |r-s|\,.$$

Then the convolution kernel $\langle r\rangle^{-2}$ is bounded on $L^2(M(r)dr)$ by a uniform constant.

Proof. We need to show that the integral kernel

$$\frac{M(r)\,M(s)^{-1}}{\langle r-s\rangle^2}=\frac{M(r)-M(s)}{\langle r-s\rangle^2}\,M(s)^{-1}+\frac{1}{\langle r-s\rangle^2}$$

is bounded on $L^2(dr)$. Since $M(s)^{-1}\leq 1$ and the latter kernel is integrable, it suffices to show that the map

$$f\;\to\;\int_{-\infty}^{\infty}\frac{M(r)-M(s)}{\langle\,r-s\,\rangle^2}\,f(s)\,ds$$

is bounded on $L^2(dr)$. Clearly

$$f\;\to\;\int_{|r-s|\leq 1}\frac{M(r)-M(s)}{\langle\,r-s\,\rangle^2}\,f(s)\,ds$$

is bounded on $L^2(dr)$, and so it suffices to show that

$$\left\|\int_{|r-s|>1}\frac{M(r)-M(s)}{\langle\,r-s\,\rangle^2}\,f(s)\,ds\right\|_{L^2(dr)}\leq C\|f\|_{L^2(dr)}. \tag{6.25}$$

But

$$\left|\,\frac{M(r)-M(s)}{\langle\,r-s\,\rangle^2}-\frac{M(r)-M(s)}{(r-s)^2}\,\right|\leq\frac{|M(r)-M(s)|}{\langle\,r-s\,\rangle^2(r-s)^2}\leq\frac{1}{|r-s|^3},$$

which means that (6.25) holds if and only if the map

$$f\;\to\;\int_{|r-s|>1}\frac{M(r)-M(s)}{(r-s)^2}\,f(s)\,ds$$

is bounded on $L^2(dr)$. But since M is Lipschitz, this follows from the classical commutator estimate of Calderón (Theorem 2 in [1]). ☐

In the following steps, we will use r and s as real variables that take the place of x_1.

Let $J_{r,s}:\mathbb{R}^4\to\mathbb{R}^4$ denote the flow along D_1-D_q, starting at the slice $x_1=s$ and ending at $x_1=r$. We will also use $J_{r,s}$ to denote the unitary map on $L^2(\mathbb{R}^4)$

$$\big(J_{r,s}f\big)(x',\xi')=f\big(J_{s,r}(x',\xi')\big)\,.$$

This map is unitary since the Hamiltonian flow is symplectic, hence preserves $dx'\,d\xi'$. We also use the fact that, if $|\xi_2|\,,\,|\eta_2|\approx\mu\theta_j$ and $|\xi_3|\,,\,|\eta_3|\approx\mu$, then the map $J_{r,s}$ approximately preserves dist_μ, in that

$$\mathrm{dist}_\mu\big(J_{r,s}(x',\xi');J_{r,s}(y',\eta')\big)\approx\mathrm{dist}_\mu(x',\xi';y',\eta')\,.$$

By homogeneity of the Hamiltonian flow, this follows from the fact that the flow is Lipschitz on the set $|\xi'|=1$, which is a consequence of Theorem 5.1.

The function $\tilde v$ satisfies

$$D_1\tilde v - D_q\tilde v = \psi_j K\tilde v\,, \qquad \tilde v(0,x',\xi') = \tilde f(x',\xi')\,.$$

Let $\mathcal U$ denote the map, taking the space of functions on $\tilde S$ to itself, defined by

$$\mathcal U F(r,\,\cdot\,) = \int_0^r J_{r,s}\psi_j K F(s,\,\cdot\,)\,ds\,.$$

Thus, $(D_1 - D_q)\mathcal U F = \psi_j K F$. If we let $F(r,\,\cdot\,) = J_{r,0}\tilde f$, so that $D_1 F - D_q F = 0$, then we can formally write the solution $\tilde v$ as

$$\tilde v = \sum_{n=0}^{\infty} \mathcal U^n F\,.$$

We need to show this sum converges in the appropriate norm, which we do by showing that $\mathcal U$ is a contraction. We split $\mathcal U = \mathcal U_1 + \mathcal U_2$, corresponding to the splitting $K = K_1 + K_2$.

The estimates we require for $\mathcal U_1$ are:

$$\|M\mathcal U_1 F\|_{L^\infty L^2(\tilde S)} \lesssim \varepsilon\, \|MF\|_{L^\infty L^2(\tilde S)} \tag{6.26}$$

$$(\mu^{\frac12}\theta_j)^{\frac12}\, \|M\langle \mu^{\frac12} x_2\rangle^{-1}\mathcal U_1 F\|_{L^2(\tilde S)} \lesssim \varepsilon\, \|MF\|_{L^\infty L^2(\tilde S)}\,. \tag{6.27}$$

For the $\mathcal U_2$ term we require the bounds:

$$\|M\mathcal U_2 F\|_{L^\infty L^2(\tilde S)} \lesssim c\,(\mu^{\frac12}\theta_j)^{\frac12}\, \|M\langle \mu^{\frac12} x_2\rangle^{-1} F\|_{L^2(\tilde S)} \tag{6.28}$$

$$\|M\langle \mu^{\frac12} x_2\rangle^{-1}\mathcal U_2 F\|_{L^2(\tilde S)} \lesssim c\, \|M\langle \mu^{\frac12} x_2\rangle^{-1} F\|_{L^2(\tilde S)} \tag{6.29}$$

The inequality (6.26) is a consequence of Lemma 6.1 with $k = n = N = 0$, and the fact that $J_{r,s}$ preserves the distance function dist_μ, hence the weight M.

For (6.27), we apply Cauchy-Schwarz to yield

$$|M\mathcal U_1 F|^2(r,x',\xi') \lesssim \varepsilon \int_0^\varepsilon \bigl|(M\psi_j K_1 F)(s, J_{s,r}(x',\xi'))\bigr|^2\,ds\,.$$

We multiply by $\langle \mu^{\frac12} x_2\rangle^{-2}$ and integrate $dx'\,d\xi'$, changing variables by $J_{s,r}$ on the right, to obtain

$$\|M\langle \mu^{\frac12} x_2\rangle^{-1}\mathcal U_1 F\|^2_{L^2(\tilde S)} \lesssim \varepsilon \int_0^\varepsilon\int_0^\varepsilon\int_{\mathbb R^4} \langle \mu^{\frac12} x_2\circ J_{r,s}\rangle^{-2}\bigl|M\psi_j K_1 F\bigr|^2(s,x',\xi')\,dx'\,d\xi'\,ds\,dr\,.$$

We next observe that, for ξ' in the support of ψ_j,

$$\int \langle \mu^{\frac12} x_2\circ J_{r,s}\rangle^{-2}\,dr \lesssim (\mu^{\frac12}\theta_j)^{-1}\,, \tag{6.30}$$

which holds since $\frac{dx_2}{dr}\approx\theta_j$. Lemma 6.1 with $k = n = N = 0$ now yields (6.27).

To show (6.28), we write

$$|(M\mathcal{U}_2 F)(r,x',\xi')|^2 \lesssim \left|\int_0^r \bigl(M\psi_j K_2 F\bigr)(s, J_{s,r}(x',\xi'))\,ds\right|^2$$
$$\lesssim (\mu^{\frac12}\theta_j)^{-1}\int_0^\varepsilon \bigl|M\langle \mu^{\frac12}x_2\rangle \psi_j K_2 F\bigr|^2(s, J_{s,r}(x',\xi'))\,ds$$

where we use (6.30). To conclude (6.27) we take the integral $dx'\,d\xi'$ of both sides, using the fact that $J_{s,r}$ preserves the measure, and apply Lemma 6.1 with $k=1$, $n=0$, and $N=2$.

For (6.29), we write as above

$$|(M\mathcal{U}_2 F)(r,x',\xi')|^2 \lesssim (\mu^{\frac12}\theta_j)^{-1}\int_0^\varepsilon \bigl|M\langle \mu^{\frac12}x_2\rangle \psi_j K_2 F\bigr|^2(s, J_{s,r}(x',\xi'))\,ds\,.$$

For ξ' in the support of ψ_j we have

$$\langle \mu^{\frac12}x_2\rangle^{-2}\,\langle \mu^{\frac12}x_2\circ J_{s,r}\rangle^{-2} \lesssim \langle \mu^{\frac12}\theta_j|r-s|\,\rangle^{-2}\,,$$

and consequently

$$\langle\, \mu^{\frac12}x_2\rangle^{-2}\,|(M\mathcal{U}_2 F)(r,x',\xi')|^2$$
$$\lesssim (\mu^{\frac12}\theta_j)^{-1}\int_0^\varepsilon \langle\, \mu^{\frac12}\theta_j|r-s|\,\rangle^{-2}\bigl|M\langle \mu^{\frac12}x_2\rangle^2 \psi_j K_2 F\bigr|^2(s, J_{s,r}(x',\xi'))\,ds\,.$$

We take the integral $dx'\,d\xi'$, changing variables by $J_{s,r}$ on the right, and apply Lemma 6.1 with $k=2$, $n=0$ and $N=3$, to yield (6.29).

We now turn to the proof of (6.22)–(6.24). First, note that by (6.26)–(6.27) and (6.28)–(6.29), for small c and ε the map $\mathcal{U}$ is a contraction in the norm

$$|||F||| = \|MF\|_{L^\infty L^2(\tilde S)} + \mu^{\frac14}\theta_j^{\frac12}\,\|M\langle \mu^{\frac12}x_2\rangle^{-1}F\|_{L^2(\tilde S)}\,.$$

Recall that $\tilde v = \sum_{n=0}^\infty \mathcal{U}^n F$, where $F(r,\,\cdot\,) = T_{r,0}\tilde f$. Furthermore,the bound (6.30) yields

$$|||F||| \lesssim \|M\tilde f\|_{L^2}\,.$$

Consequently

$$|||\tilde v||| \lesssim \|M\tilde f\|_{L^2}\,,$$

which implies (6.22) and (6.23).

To derive (6.24), we use the fact that each of the estimates (6.26)–(6.29) holds if M is in each instance replaced by the weight

$$M\langle \mu^{\frac12}(x_2 - c_2\theta_j x_1)_-\rangle\,,$$

where $c_2>0$ is a constant such that $\frac{dx_2}{dr} > c_2\theta_j$ on curves of $D_1 - D_q$ passing through the support of ψ_j. This holds since $\langle \mu^{\frac12}(\,\cdot\,)_-\rangle$ is positive and decreasing, and hence, if $\psi_j(\xi')\neq 0$ and $r\ge s$, then

$$\langle \mu^{\frac12}(x_2\circ J_{r,s} - c_2\theta_j r)_-\rangle \le \langle \mu^{\frac12}(x_2 - c_2\theta_j s)_-\rangle\,.$$

Consequently,

$$\begin{aligned}\|M\langle\mu^{\frac12}(x_2-c_2\theta_jx_1)_-\rangle\langle\mu^{\frac12}x_2\rangle^{-1}\tilde v\|_{L^2(\tilde S)} &\lesssim (\mu^{\frac12}\theta_j)^{-\frac12}\|M\langle\mu^{\frac12}(x_2)_-\rangle\tilde f\|_{L^2}\\ &\lesssim (\mu^{\frac12}\theta_j)^{-\frac12}\|M\langle\mu^{\frac12}x_2\rangle\tilde f\|_{L^2}\,.\end{aligned}$$

On the other hand, for $x_1>0$,

$$\langle\mu^{\frac12}(x_2-c_2\theta_jx_1)_-\rangle\,\langle\mu^{\frac12}x_2\rangle\gtrsim\langle\mu^{\frac12}\theta_jx_1\rangle\,,$$

hence

$$\|M\langle\mu^{\frac12}x_2\rangle^{-2}\tilde v\|_{L^1L^2(\tilde S)}\lesssim(\mu^{\frac12}\theta_j)^{-\frac12}\|M\langle\mu^{\frac12}(x_2-c_2\theta_jx_1)_-\rangle\langle\mu^{\frac12}x_2\rangle^{-1}\tilde v\|_{L^2(\tilde S)}\,,$$

yielding (6.24).

6.3. **The estimate on K.** We establish here the estimate (6.21) for the integral kernel K defined by

$$T_\mu Q_\mu T_\mu^*=D_q+K\,.$$

Here, $Q_\mu=\frac12\bigl(q_\mu(x,D')+q_\mu(x,D')^*\bigr)$, where q_μ is the symbol p_λ rescaled by θ_j, and hence truncated to x'-frequencies less than $c\mu$. The symbol q, on the other hand, is obtained by truncating q_μ to x'-frequencies less than $c\mu^{\frac12}$.

It is a simple consequence of Lemma 4.3 and Lemma 4.2 that the kernel of the operator

$$T_\mu\,q(x,D')^*\,T_\mu^*-D_q$$

satisfies the estimate (6.21). By taking adjoints the same applies with $q(x,D')^*$ replaced by $q(x,D')$, and we are reduced to establishing the estimates (6.21) for the kernel of the operator

$$T_\mu\,(q_\mu(x,D')-q(x,D')\,)\,T_\mu^*\,.$$

The kernel $K(x',\xi';y',\eta')$ of this operator takes the form (we suppress the irrelevant parameter x_1)

$$\int e^{i\langle\zeta',z'-y'\rangle}\,e^{-i\langle\xi',z'-x'\rangle}\Bigl[q_\mu(z',\zeta')-q(z',\zeta')\Bigr]\,\widehat g\bigl(\mu^{-\frac12}(\zeta'-\eta')\bigr)\,g\bigl(\mu^{\frac12}(z'-x')\bigr)\,dz'\,d\zeta'\,.$$

Suppose that $p(x')$ is a smooth function on $x_2\ge0$, which is constant for $x_2\ge1$. We extend p in an even manner to $x_2\le0$. Let $q_\mu=S_\mu[p(\theta_j\,\cdot)]$, and $q=S_{\sqrt\mu}[p(\theta_j\,\cdot)]$, where S_λ denotes smooth truncation of the Fourier transform to frequencies less than $c\lambda$. It then follows that

$$\begin{aligned}&\bigl|D^\beta_{x'}\bigl(q_\mu(x')-q(x')\bigr)\bigr|\lesssim\theta_j\mu^{\frac12(|\beta|-1)}\langle\mu^{\frac12}x_2\rangle^{-N}\,\|D_{x'}p\|_{C^N(x_2\ge0)}\,,\quad|\beta|\le1\,,\\ &\bigl|D^2_{x'}\bigl(q_\mu(x')-q(x')\bigr)\bigr|\lesssim\theta_j\Bigl(\mu^{\frac12}\langle\mu^{\frac12}x_2\rangle^{-N}+\mu\langle\mu x_2\rangle^{-N}\Bigr)\|D_{x'}p\|_{C^{N+2}(x_2\ge0)}\,.\end{aligned}\tag{6.31}$$

Indeed, it suffices to verify these bounds for $x_2>0$, and by splitting up p to separately consider the case that p is smooth across $x_2=0$ and constant for $|x_2|\ge1$, and the case that p is smooth on $x_2\le0$ and vanishes for $x_2\ge0$. The latter case is handle by simple size bounds on the convolution kernels S_μ and $S_{\sqrt\mu}$. For the smooth part, we have bounds

$$\begin{aligned}|p_\lambda-S_{\theta_j^{-1}\mu^{\frac12}}p_\lambda|(\theta_jx')&\lesssim(\theta_j^{-1}\mu^{\frac12})^{-1-N}\langle\theta_jx_2\rangle^{-N}\,\|D^{N+1}_{x'}p\|_{C^0}\\ &\lesssim\theta_j\,\mu^{-\frac12}\langle\mu^{\frac12}x_2\rangle^{-N}\,\|D^{N+1}_{x'}p\|_{C^0}\,,\end{aligned}$$

and the same bounds apply to derivatives.

By the condition (2.3), we easily obtain the following bounds for $|\zeta'| \approx \mu$ and $\beta_2 \le 2$,

$$\left|\partial_{z'}^{\beta}\partial_{\zeta'}^{\alpha}\big(\, q_\mu(z',\zeta') - q(z',\zeta')\big)\right| \lesssim c_0\,\theta_j \Big(\mu^{\frac12}\langle \mu^{\frac12} z_2\rangle^{-N} + \mu\langle \mu z_2\rangle^{-N}\Big)\mu^{\frac12|\beta_2|-|\alpha|}\,.$$

In the formula for K we can integrate by parts at will with respect to $\mu^{\frac12}D_{\zeta'}$ and $\mu^{-\frac12}D_{z_3}$, and twice with respect to $\mu^{-\frac12}D_{z_2}$, to dominate K by

$$c_0\,\theta_j \int \Big(\mu^{\frac12}\langle \mu^{\frac12} z_2\rangle^{-N} + \mu\langle \mu z_2\rangle^{-N}\Big)\langle \mu^{\frac12}(z'-x')\rangle^{-N}\langle \mu^{\frac12}(z'-y')\rangle^{-N} \\ \times \langle \mu^{-\frac12}(\zeta'-\eta')\rangle^{-N}\langle \mu^{-\frac12}(\zeta_3-\xi_3)\rangle^{-N}\langle \mu^{-\frac12}(\zeta_2-\xi_2)\rangle^{-2}\,dz'\,d\zeta'$$

which is dominated by

$$c_0\,\theta_j\,\mu^{\frac12}\langle \mu^{\frac12}x_2\rangle^{-N}\,\langle \mu^{\frac12}(x'-y')\rangle^{-N}\,\langle \mu^{-\frac12}(\xi_3-\eta_3)\rangle^{-N}\,\langle \mu^{-\frac12}(\xi_2-\eta_2)\rangle^{-2}$$

yielding the desired bounds on K. □

We note here that similar considerations to the above yield the bounds, for $|\xi'| \approx \mu$,

$$\left|\partial_x^{\beta}\partial_{\xi'}^{\alpha} q(x,\xi')\right| \lesssim \begin{cases} \mu^{1-|\alpha|}\,, & |\beta| = 0\,,\\ c_0\,\big(\,1 + \mu^{\frac12(|\beta|-1)}\theta_j\,\langle \mu^{\frac12}x_2\rangle^{-N}\,\big)\,\mu^{1-|\alpha|}\,, & |\beta| \ge 1\,. \end{cases} \tag{6.32}$$

6.4. **Estimates on F_j and G_j.** We conclude by establishing the square summability of the inhomogeneities F_j and G_j. Recall that

$$F_j = \beta_j(D')F_\lambda + [\beta_j(D'),P_j]u_\lambda + \beta_j(D')\big(P_\lambda - p_\lambda(x,D')\big)u_\lambda\,,$$
$$G_j = \beta_j(D')\big(p_\lambda(x,D') - p_j(x,D')\big)u_\lambda\,.$$

We need to show that

$$\sum_j \|F_j\|^2_{L^1L^2(S_j)} + \lambda^{-\frac14}\theta_j^{-\frac14}\|\langle \lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^2 G_j\|^2_{L^2(S_j)} \lesssim \|u_\lambda\|^2_{L^\infty L^2(S)} + \|F_\lambda\|^2_{L^2(S)}\,.$$

The first term in F_j is handled by noting that

$$\sum_j \|\,\beta_j(D')F_\lambda\|^2_{L^1L^2(S_j)} \le \sum_j \|\,\beta_j(D')F_\lambda\|^2_{L^2(S)} \le \|F_\lambda\|^2_{L^2(S)}\,.$$

Consider next the term $[P_j,\beta_j(D')]u_\lambda$. Since the symbol of P_j is truncated to x'-frequencies less than $c\mu^{\frac12} \le c\,\theta_j\mu$, it holds that

$$[P_j,\beta_j(D')]u_\lambda = [P_j,\beta_j(D')]\,\phi_j(D')u_\lambda\,.$$

We claim that, uniformly over x_1,

$$\big\|[P_j,\beta_j(D')]f\big\|_{L^2_{x'}} \lesssim 2^j\,\|f\|_{L^2_{x'}}\,. \tag{6.33}$$

Given this, we can bound

$$\big\|[P_j,\beta_j(D')]\phi_j(D')u_\lambda\big\|_{L^1L^2(S_j)} \lesssim 2^j\,\|\,\phi_j(D')u_\lambda\|_{L^1L^2(S_j)} \le \|\,\phi_j(D')u_\lambda\|_{L^\infty L^2(S_j)}\,,$$

since S_j is of length 2^{-j} in x_1. Since $\phi_j(D')u_\lambda$ involves $\beta_j(D')u_\lambda$ only for $|i-j| \le 4$, this term is square summable over j.

To prove (6.33), it suffices to replace P_j by $p_j(x,D')$. The symbol p_j equals $|\xi'|$ outside of the region $|\xi'|\approx\lambda$, and $p_j(x',\xi')$ satisfies $S^1_{1,0}$ estimates for x' derivatives of order at most 1. Consquently, after subtracting off the term $|D'|$, we may take $p_j(x,D')$ to have kernel $K_1(x',x'-y')$ where

$$|K_1(x',z')|+|D'_xK_1(x',z')|\lesssim \lambda\cdot\lambda^2\,\bigl(1+\lambda|z'|\bigr)^{-N}\,.$$

On the other hand, $\beta_j(D')$ is a convolution kernel $K_2(x'-y')$ where

$$\|z'\,K_2(z')\|_{L^1_{z'}}\lesssim 2^j\lambda^{-1}\,.$$

The estimate (6.33) follows by applying Taylor's theorem to

$$\bigl[K_1,K_2\bigr](x',y')=\int K_2(x'-z')\bigl(K_1(x',z'-y')-K_1(z',z'-y')\bigr)\,dz'\,.$$

To control the last term in F_j we note that, since the estimates on K_1 above also apply to $p_\lambda(x,D')$, we have uniform bounds

$$\bigl\|\bigl(P_\lambda-p_\lambda(x,D')\bigr)f\bigr\|_{L^2_{x'}}=\tfrac12\bigl\|\bigl(p_\lambda(x,D')^*-p_\lambda(x,D')\bigr)f\bigr\|_{L^2_{x'}}\lesssim\|f\|_{L^2_{x'}}\,.$$

The last term in F_j is orthogonal over j, and thus has square sum bounded by $\|u\|_{L^2(S)}$.

We now estimate the term G_j. We split this up

$$G_j=\beta_j(D')\bigl(p_\lambda(x,D')-p_j(x,D')\bigr)\phi_j(D')u_\lambda+\beta_j(D')p_\lambda(x,D')\bigl(1-\phi_j(D')\bigr)u_\lambda\,.$$

Consider the second term in G_j. We write

$$\beta_j(D')p_\lambda(x,D')\bigl(1-\phi_j(D')\bigr)u_\lambda=\beta_j(D')\bigl(p_\lambda(x,D')-p_{\lambda\theta_j}(x,D')\bigr)\bigl(1-\phi_j(D')\bigr)u_\lambda$$

where $p_{\lambda\theta_j}$ is the symbol p truncated to x'-frequencies of size less than $c\lambda\theta_j$. The symbol $p_\lambda-p_{\lambda\theta_j}$ is supported in the region $|\xi'|\approx\lambda$, and by arguments similar to those deriving (6.31) (without the rescaling step), we have the estimates

$$|\partial^\alpha_{\xi'}(p_\lambda-p_{\lambda\theta_j})(x,\xi')|\lesssim\theta_j^{-1}\langle\lambda\theta_jx_2\rangle^{-N}\,\lambda^{-|\alpha|}\,.$$

Its integral kernel is thus bounded by

$$\theta_j^{-1}\langle\lambda\theta_jx_2\rangle^{-N}\,\lambda^2\,\bigl(1+\lambda\,|x'-y'|\,\bigr)^{-N}\,.$$

Since $\lambda\theta_j\ge\lambda^{\frac12}\theta_j^{-\frac12}$, it follows that, uniformly in x_1,

$$\lambda^{-\frac14}\theta_j^{-\frac14}\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^2\bigl(p_\lambda(x,D')-p_{\lambda\theta_j}(x,D')\bigr)f\|_{L^2_{x'}}\le\lambda^{-\frac14}\theta_j^{-\frac54}\|f\|_{L^2_{x'}}\,,$$

and the same holds for $p_\lambda(x,D')-p_{\lambda\theta_j}(x,D')$ replaced by $\beta_j(D')\bigl(p_\lambda(x,D')-p_{\lambda\theta_j}(x,D')\bigr)$ since $\beta_j(D')$ averages on scale smaller than $\lambda^{-\frac12}\theta_j^{\frac12}$. Thus

$$\lambda^{-\frac14}\theta_j^{-\frac14}\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^2\beta_j(D')p_\lambda(x,D')\bigl(1-\phi_j(D')\bigr)u_\lambda\|_{L^2(S_j)}\le\lambda^{-\frac14}2^{\frac34j}\|u_\lambda\|_{L^\infty L^2(S)}\,.$$

Since 2^j runs from 1 to $\lambda^{\frac13}$, the right hand side is square summable over j.

Recalling that the symbol $p_j(x,\xi')$ is truncated to x'-frequencies less than $\lambda^{\frac12}\theta_j^{-\frac12}$, similar arguments show that

$$\begin{aligned}\lambda^{-\frac14}\theta_j^{-\frac14}\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^2\beta_j(D')\bigl(p_\lambda(x,D')-p_j(x,D')\bigr)\phi_j(D')u_\lambda\|_{L^2(S_j)}&\\ \lesssim\lambda^{\frac14}\theta_j^{\frac14}\|\langle\lambda^{\frac12}\theta_j^{-\frac12}x_2\rangle^{-1}\phi_j(D')u_\lambda\|_{L^2(S_j)}\,.&\end{aligned}$$

The right hand side involves u_i for $|i-j|\le 4$, hence, by the earlier estimate for u_i, is square summable over j.

7. Results for higher dimensions

We show here that the steps of the preceeding sections yield sharp L^q estimates for spectral clusters on compact Riemannian manifolds M with boundary, of dimension $n\ge 3$, provided q is sufficiently large. Precisely, we have the following

Theorem 7.1. *Suppose that u solves the Cauchy problem on $\mathbb{R}\times M$*

$$\partial_t^2 u(t,x)=Pu(t,x)\,,\qquad u(0,x)=f(x)\,,\qquad \partial_t u(0,x)=0\,,$$

and satisfies either Dirichlet conditions

$$u(t,x)=0 \quad \textit{if} \quad x\in\partial M\,,$$

or Neumann conditions, where N_x is a unit normal field with respect to g,

$$N_x\cdot\nabla_x u(t,x)=0 \quad \textit{if} \quad x\in\partial M\,.$$

Then the following bounds hold for $4\le q\le\infty$, if $n\ge 4$, and $5\le q\le\infty$ if $n=3$.

$$\|u\|_{L^q_x L^2_t(M\times[-1,1])}\le C\,\|f\|_{H^{\delta(q)}(M)}\,,\qquad \delta(q)=n\big(\tfrac12-\tfrac1q\big)-\tfrac12\,.$$

These bounds of course imply that the estimates in (1.9) hold for the spectral projector operators, χ_λ, when $q\ge 5$ for $n=3$ and $q\ge 4$ if $n\ge 4$.

As noted in the introduction, these estimates are expected to hold in the larger range $q\ge\frac{6n+4}{3n-4}$, in which case they (and their interpolation with the trivial L^2 estimate) would be best possible. Establishing this larger range would require exploiting dispersion in directions tangent to ∂M for time 1, rather than times on the order of the microlocalization angle θ.

Following the earlier sections, we work in a neighborhood of ∂M in geodesic normal coordinates, and extend the operator P evenly, and solution u oddly or evenly, in the case of Dirichlet or Neumann conditions respectively. We set $x_{n+1}=t$, and $x'=(x_2,\dots,x_{n+1})$.

We then fix a frequency scale λ and microlocalization angle $\theta_j\in[\lambda^{-\frac13},c]$. After factorizing $DA_\lambda D$, we set

$$q(x,\xi')=\theta_j\,p_j(\theta_j x,\theta_j^{-1}\xi')\,,$$

which is x'-frequency localized at scale $\mu^{\frac12}$, where $\mu=\theta\lambda$ is the frequency scale of the rescaled solution $u(\theta x)$ (we suppress the index j.) We work with the wave packet transform $\tilde u$ of u with respect to the x' variables, and let Θ denote the Hamiltonian flow along $\xi_1-q(x,\xi')$. The reduction steps of sections 2 through 4 can then be adapted to reduce matters to establishing the following.

Theorem 7.2. *Suppose that $f\in L^2(\mathbb{R}^{2n})$ is supported in a set of the form $\xi_{n+1}\approx\mu$, $|\xi_2,\dots,\xi_{n-1}|\le c\mu$, $\xi_n\approx\theta\mu$ or $|\xi_n|\lesssim\mu^{\frac12}$ in case $\theta=\mu^{-\frac12}$.*

Then, if $u=T_\mu^*\big[f\big(\Theta_{0,x_1}(x',\xi')\big)\big]$, we have for $q\ge\frac{2n}{n-2}$

$$\|u\|_{L^qL^2(S)}\lesssim\mu^{\delta(q)}\theta^{\frac12-\frac1q}\,\|f\|_{L^2(\mathbb{R}^{2n})}\,,$$

and for $\frac{2(n+1)}{n-1} \le q \le \frac{2n}{n-2}$

$$\|u\|_{L^q L^2(S)} \lesssim \mu^{\delta(q)} \theta^{(n-1)(\frac12-\frac1q)-\frac2q} \, \|f\|_{L^2(\mathbb{R}^{2n})} \,.$$

This implies Theorem 7.1 for q such that the exponent of θ is at least $\frac1q$. For $n \ge 4$, this happens for $q \ge 4 \ge \frac{2n}{n-2}$. For $n = 3$ this holds for $q \ge 5$.

In applying the reductions of section 2, care must be taken since $\delta(q) \ge 1$ for large n, whereas the commutator $[A, \Gamma(D)]$ maps $H^{\delta-1} \to H^\delta$ only for $0 \le \delta \le 1$. Here, $\Gamma(D)$ is a conic cutoff to the set $|\xi_{n+1}| \approx |\xi_1, \dots, \xi_n|$. To get around this problem, in case $\delta(q) \ge 1$ we write $\delta(q) = m + \delta$, with $0 \le \delta \le 1$. Let $d_T = (d_1, \dots, d_{n-1}, d_{n+1})$ denote the tangential derivatives, and d_T^m the collection of tangential derivatives of order at most m. Then the extended and ϕ-localized solution u satisfies

$$\|d_T^m u\|_{H^\delta} + \|d_T^m F\|_{H^\delta} \lesssim \|f\|_{H^{\delta(q)}(M)} \,.$$

Since $d_T^m A$ is Lipschitz, it is easy to see that

$$\|d_T^m [A, \Gamma(D)] D u\|_{H^\delta} \lesssim \|d_T^m u\|_{H^\delta} \,.$$

We also gain powers of d_T^m in the elliptic regularity arguments, and deduce that

$$\|d_T^m (1 - \Gamma(D)) u\|_{H^{\delta+1}} \lesssim \|d_T^m u\|_{H^\delta} + \|d_T^m F\|_{H^\delta} \,.$$

The norm on the left is sufficient to control $\|(1-\Gamma(D))u\|_{L^q_x L^2_t}$, and we are reduced to considering $\Gamma(D)u$. This term, however, has Fourier transform supported outside of a conic neighborhood of the ξ_n axis, hence

$$\|\Gamma(D) u\|_{H^{\delta(q)}} \approx \|d_T^m \Gamma(D) u\|_{H^\delta} \,.$$

The remaining reductions of section 2 then follow.

To prove Theorem 7.2, we establish mapping properties for the kernel K of WW^*, localized in $\zeta = (\zeta_2, \dots, \zeta_{n+1})$ by a cutoff $\beta_\theta(\zeta)$ to the set

$$\zeta_{n+1} \approx \mu \,, \qquad |(\zeta_2, \dots, \zeta_{n-1})| \le c\,\mu \,, \qquad \zeta_n \approx \theta\mu \,,$$

(respectively $|\zeta_n| \le \mu^{-\frac12}$ in case $\theta = \mu^{-\frac12}$.) The bounds we establish, analogous to (5.3) and (5.4), are

$$\sup_{r,s \in [0,\varepsilon]} \Bigl\| \int K(r, x'; s, y')\, f(y')\, dy' \Bigr\|_{L^2_{x'}} \le \|f\|_{L^2_{y'}} \,, \tag{7.1}$$

and

$$\begin{aligned} &\Bigl\| \int K(r, x'; s, y')\, f(y')\, dy \Bigr\|_{L^\infty_{x_2,\dots,x_n} L^2_{x_{n+1}}} \\ &\qquad \lesssim \mu^{n-1} \theta \bigl(1 + \mu\,|r-s| \bigr)^{-\frac{n-2}{2}} \bigl(1 + \mu\theta^2\,|r-s| \bigr)^{-\frac12} \|f\|_{L^1_{y_2,\dots,y_n} L^2_{y_{n+1}}} \,. \end{aligned} \tag{7.2}$$

To see that this implies Theorem 7.2, note that interpolation yields the bound

$$\begin{aligned} &\Bigl\| \int K(r, x'; s, y')\, f(y')\, dy \Bigr\|_{L^q_{x_2,\dots,x_n} L^2_{x_{n+1}}} \\ &\lesssim \bigl(\mu^{n-1}\theta \bigr)^{1-\frac2q} \bigl(1 + \mu|r-s| \bigr)^{-(n-2)(\frac12-\frac1q)} \bigl(1 + \mu\theta^2|r-s| \bigr)^{-(\frac12-\frac1q)} \|f\|_{L^{q'}_{y_2,\dots,y_n} L^2_{y_{n+1}}} \,. \end{aligned}$$

If $q \ge \frac{2n}{n-2}$, then $(n-2)(\frac12-\frac1q)\ge\frac2q$, and by the Hardy-Littlewood-Sobolev lemma we obtain

$$\Bigl\|\int K(r,x';s,y')\,F(s,y')\,dy\Bigr\|_{L^q_{r,x_2,\dots,x_n}L^2_{x_{n+1}}}\lesssim \mu^{2\delta(q)}\theta^{1-\frac2q}\,\|F\|_{L^{q'}_{s,y_2,\dots,y_n}L^2_{y_{n+1}}}\,.$$

If $\frac{2n}{n-2}\ge q\ge\frac{2(n+1)}{n-1}$, then

$$\bigl(\,1+\mu|r-s|\,\bigr)^{-(n-2)(\frac12-\frac1q)}\bigl(\,1+\mu\theta^2|r-s|\,\bigr)^{-(\frac12-\frac1q)}\le\mu^{-\frac2q}\theta^{-\frac4q+(n-2)(1-\frac2q)}\,|r-s|^{-\frac2q}\,,$$

and Hardy-Littlewood-Sobolev yields Theorem 7.2 for this case.

We now turn to the proof of (7.1) and (7.2). The estimate (7.1) follows as does the estimate (5.3) from the boundedness of T_μ and the fact that $\Theta_{r,s}$ preserves the measure $dx'\,d\xi'$. To establish (7.2), we consider as before separate cases, depending on $|r-s|$.

Consider the case $\mu\theta^2|r-s|\ge1$. We fix $\bar\theta\le\theta$ so that $\mu\bar\theta^2|r-s|=1$, and decompose $\beta_\theta(\zeta)$ into a sum of cutoffs $\beta_j(\zeta)$, each of which is localized to a cone of angle $\bar\theta$ about some direction. As in the proof of (5.16), we have that

$$\int|K_j(r,x';s,y')|\,dy_{n+1}\,\lesssim\,\mu^{n-1}\bar\theta^{n-1}\bigl(\,1+\mu\bar\theta|(y'-w^j_{s,r})_{2,\dots,n}|\bigr)^{-N}\,,$$

where the $w^j_{s,r}$ give a $(\mu\bar\theta)^{-1}$ separated set after projection onto the $(2,\dots,n)$ variables. Adding over j yields the desired bounds, since

$$\mu^{n-1}\bar\theta^{n-1}=\mu^{\frac{n-1}{2}}\,|r-s|^{-\frac{n-1}{2}}\,.$$

In case $\mu\theta^2|r-s|\le1$, let $\bar\theta\ge\theta$ be given by

$$\bar\theta=\min\bigl(\,\mu^{-\frac12}|r-s|^{-\frac12}\,,\,1\,\bigr)\,.$$

We set $\zeta''=(\zeta_2,\dots,\zeta_{n-1},\zeta_{n+1})$, and let β_j be a partition of unity in cones of angle $\bar\theta$ on $\mathbb{R}^{n-1}$. We then decompose

$$\beta_\theta(\zeta)=\sum_j\beta_\theta(\zeta)\,\beta_j(\zeta'')\,.$$

Let $K=\sum_jK_j$ denote the corresponding kernel decomposition. As in the proof of Theorem 5.4, we can bound K_j by

$$\mu^{\frac n2}\int\bigl(\,1+\mu\bar\theta\,|d_{\zeta''}\zeta_{s,r}\cdot(y'-x'_{s,r})|+\mu\theta\,|d_{\zeta_n}\zeta_{s,r}\cdot(y'-x'_{s,r})|+|\langle\zeta_{s,r},y'-x'_{s,r}\rangle|\,\bigr)^{-N}\\ \times\bigl(\,1+\mu^{\frac12}|x'-z|\,\bigr)^{-N}\,dz\,d\zeta\,.$$

Here, $(x'_{s,r},\xi'_{s,r})=\Theta_{s,r}(x',\xi'_j)$, with ξ'_j a fixed vector in the support of $\beta_\theta(\zeta)\beta_j(\zeta'')$. Also, $(z_{s,r},\zeta_{s,r})=\Theta_{s,r}(z,\zeta)$. Since $d_\zeta\zeta_{s,r}$ is invertible, and $\mu\bar\theta\ge\mu\theta\ge\mu^{\frac12}$, the first two terms in the integrand dominate $\mu^{\frac12}|y'-x'_{s,r}|$.

We first show that we may replace $\zeta_{s,r}$ by $\xi'_{s,r}=\zeta_{s,r}(x',\xi'_j)$ in the third term in parentheses above. By homogeneity, we may consider $|\zeta|=|\xi'_j|$. We take a first order Taylor expansion, and use bounds (5.7) on $d^2_\zeta\zeta_{s,r}$, to write

$$\zeta_{s,r}-\zeta_{s,r}(z,\xi'_j)=(\zeta-\xi'_j)\cdot d_\zeta\zeta_{s,r}+O(\,|\zeta-\xi'_j|^2\mu^{-\frac12}|s-r|\,)\,.$$

Since

$$|(\zeta-\xi'_j)''|\lesssim\mu\bar\theta\,,\qquad|(\zeta-\xi'_j)_n|\lesssim\mu\theta\,,\qquad\mu\bar\theta^2|s-r|\le1\,,$$

this shows we may replace $\zeta_{s,r}$ by $\zeta_{s,r}(z,\xi'_j)$, as the errors are absorbed by the first two terms in parentheses. On the other hand, by (5.5)

$$|\langle \zeta_{s,r}(x',\xi'_j) - \zeta(z,\xi'_j)\,, y' - x'_{s,r}\rangle\,| \lesssim \mu\,|x'-z|\,|y'-x'_{s,r}|\,,$$

which is also absorbed by the other terms.

We next use (5.5) to see that we may replace $d_\zeta \zeta_{s,r}$ by the identity matrix, since the error induced is dominated by

$$\mu\bar\theta|s-r|\,|y'-x'_{s,r}| \le \mu^{\frac12}|y'-x'_{s,r}|\,.$$

Consequently, since $\xi'_{s,r}$ has $n+1$ component comparable to μ, we obtain

$$\int |K_j(r,x';s,y')|\,dy_{n+1} \lesssim \mu^{n-1}\bar\theta^{n-2}\,\theta\,\big(\,1+\mu\bar\theta\,|(y'-x'_{s,r})_{2,\dots,n-1}|\,\big)^{-N}\,.$$

The points $x'_{s,r}$ are $\mu\bar\theta$ separated in the $(2,\dots,n-1)$ variables as j varies, which follows by Corollary 5.2 and the fact that $q(z,\zeta)$ is close to $|\zeta|$, hence we can add over j to obtain

$$\int |K(r,x';s,y')|\,dy_{n+1} \lesssim \mu^{n-1}\bar\theta^{n-2}\,\theta \lesssim \mu^{n-1}\theta\big(\,1+\mu|r-s|\,\big)^{-\frac{n-2}{2}}\,. \quad \Box$$

References

[1] A.P. Calderón, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1092–1099.

[2] R. Coifman and Y. Meyer, *Commutateurs d'integrales singulieres et operateurs multilineaires*, Ann. Inst. Fourier Grenoble **28** (1978), 177–202.

[3] A. Cordoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations **3** (1978), 979–1005.

[4] K. DeLeeuw, unpublished.

[5] D. Grieser, *L^p bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries*, Thesis, UCLA, 1992.

[6] D. Grieser, *Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary*, Comm. Partial Differential Equations, **27** (2002), 1283–1299.

[7] H. Koch and D. Tataru, *Dispersive estimates for principally normal pseudodifferential operators*, Comm. Pure Appl. Math. **58** (2005), 217–284.

[8] G. Mockenhaupt, A. Seeger and C. D. Sogge, *Local smoothing of Fourier integrals and Carleson-Sjölin estimates*, J. Amer. Math. Soc. **6** (1993), 65–130.

[9] J. W. S. Rayleigh, *The problem of the whispering gallery*, Philos. Mag. **20** (1910), 1001-1004.

[10] J. W. S. Rayleigh, *The theory of sound*, Dover, New York, 1945.

[11] H. Smith, *Spectral cluster estimates for $C^{1,1}$ metrics*, Preprint.

[12] H. Smith, *Sharp $L^2 \to L^q$ bounds on spectral projectors for low regularity metrics*, Preprint.

[13] H. Smith and C. D. Sogge, *On the critical semilinear wave equation outside convex obstacles*, J. Amer. Math. Soc. **8** (1995), 879–916.

[14] C. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–134.

[15] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Univ. Press, Cambridge, 1993.

[16] C. D. Sogge, *Eigenfunction and Bochner Riesz estimates on manifolds with boundary*, Math. Res. Lett. **9** (2002), 205-215.

[17] E. M. Stein, unpublished.

[18] D. Tataru, *Strichartz estimates for operators with nonsmooth coefficients III*, J. Amer. Math. Soc. **15** (2002), 419–442.

[19] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Progress in Mathematics, vol 100, Birkh auser, Boston, 1991.

Department of Mathematics, University of Washington, Seattle, WA 98195

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: hart@math.washington.edu

E-mail address: sogge@jhu.edu